

Chapter 10

ELECTROMAGNETIC WAVE PROPAGATION

How far you go in life depends on your being tender with the young, compassionate with the aged, sympathetic with the striving, and tolerant of the weak and the strong. Because someday in life you will have been all of these.

—GEORGE W. CARVER

10.1 INTRODUCTION

Our first application of Maxwell's equations will be in relation to electromagnetic wave propagation. The existence of EM waves, predicted by Maxwell's equations, was first investigated by Heinrich Hertz. After several calculations and experiments Hertz succeeded in generating and detecting radio waves, which are sometimes called Hertzian waves in his honor.

In general, waves are means of transporting energy or information.

Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays. All forms of EM energy share three fundamental characteristics: they all travel at high velocity; in traveling, they assume the properties of waves; and they radiate outward from a source, without benefit of any discernible physical vehicles. The problem of radiation will be addressed in Chapter 13.

In this chapter, our major goal is to solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$)
2. Lossless dielectrics ($\sigma = 0$, $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_r \mu_0$, or $\sigma \ll \omega \epsilon$)
3. Lossy dielectrics ($\sigma \neq 0$, $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_r \mu_0$)
4. Good conductors ($\sigma \approx \infty$, $\epsilon = \epsilon_0$, $\mu = \mu_r \mu_0$, or $\sigma \gg \omega \epsilon$)

where ω is the angular frequency of the wave. Case 3, for lossy dielectrics, is the most general case and will be considered first. Once this general case is solved, we simply derive other cases (1, 2, and 4) from it as special cases by changing the values of σ , ϵ , and μ . However, before we consider wave motion in those different media, it is appropriate that we study the characteristics of waves in general. This is important for proper understand-

ing of EM waves. The reader who is conversant with the concept of waves may skip Section 10.2. Power considerations, reflection, and transmission between two different media will be discussed later in the chapter.

10.2 WAVES IN GENERAL

A clear understanding of EM wave propagation depends on a grasp of what waves are in general.

A wave is a function of both space and time.

Wave motion occurs when a disturbance at point A , at time t_0 , is related to what happens at point B , at time $t > t_0$. A wave equation, as exemplified by eqs. (9.51) and (9.52), is a partial differential equation of the second order. In one dimension, a scalar wave equation takes the form of

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial z^2} = 0 \quad (10.1)$$

where u is the *wave velocity*. Equation (10.1) is a special case of eq. (9.51) in which the medium is source free ($\rho_v = 0$, $\mathbf{J} = 0$). It can be solved by following procedure, similar to that in Example 6.5. Its solutions are of the form

$$E^- = f(z - ut) \quad (10.2a)$$

$$E^+ = g(z + ut) \quad (10.2b)$$

or

$$E = f(z - ut) + g(z + ut) \quad (10.2c)$$

where f and g denote any function of $z - ut$ and $z + ut$, respectively. Examples of such functions include $z \pm ut$, $\sin k(z \pm ut)$, $\cos k(z \pm ut)$, and $e^{jk(z \pm ut)}$, where k is a constant. It can easily be shown that these functions all satisfy eq. (10.1).

If we particularly assume harmonic (or sinusoidal) time dependence $e^{j\omega t}$, eq. (10.1) becomes

$$\frac{d^2 E_s}{dz^2} + \beta^2 E_s = 0 \quad (10.3)$$

where $\beta = \omega/u$ and E_s is the phasor form of E . The solution to eq. (10.3) is similar to Case 3 of Example 6.5 [see eq. (6.5.12)]. With the time factor inserted, the possible solutions to eq. (10.3) are

$$E^+ = A e^{j(\omega t - \beta z)} \quad (10.4a)$$

$$E^- = B e^{j(\omega t + \beta z)} \quad (10.4b)$$

and

$$E = Ae^{j(\omega t - \beta z)} + Be^{j(\omega t + \beta z)} \quad (10.4c)$$

where A and B are real constants.

For the moment, let us consider the solution in eq. (10.4a). Taking the imaginary part of this equation, we have

$$E = A \sin(\omega t - \beta z) \quad (10.5)$$

This is a sine wave chosen for simplicity; a cosine wave would have resulted had we taken the real part of eq. (10.4a). Note the following characteristics of the wave in eq. (10.5):

1. It is time harmonic because we assumed time dependence $e^{j\omega t}$ to arrive at eq. (10.5).
2. A is called the *amplitude* of the wave and has the same units as E .
3. $(\omega t - \beta z)$ is the *phase* (in radians) of the wave; it depends on time t and space variable z .
4. ω is the *angular frequency* (in radians/second); β is the *phase constant* or *wave number* (in radians/meter).

Due to the variation of E with both time t and space variable z , we may plot E as a function of t by keeping z constant and vice versa. The plots of $E(z, t = \text{constant})$ and $E(t, z = \text{constant})$ are shown in Figure 10.1(a) and (b), respectively. From Figure 10.1(a), we observe that the wave takes distance λ to repeat itself and hence λ is called the *wavelength* (in meters). From Figure 10.1(b), the wave takes time T to repeat itself; consequently T is known as the *period* (in seconds). Since it takes time T for the wave to travel distance λ at the speed u , we expect

$$\lambda = uT \quad (10.6a)$$

But $T = 1/f$, where f is the *frequency* (the number of cycles per second) of the wave in Hertz (Hz). Hence,

$$u = f\lambda \quad (10.6b)$$

Because of this fixed relationship between wavelength and frequency, one can identify the position of a radio station within its band by either the frequency or the wavelength. Usually the frequency is preferred. Also, because

$$\omega = 2\pi f \quad (10.7a)$$

$$\beta = \frac{\omega}{u} \quad (10.7b)$$

and

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \quad (10.7c)$$

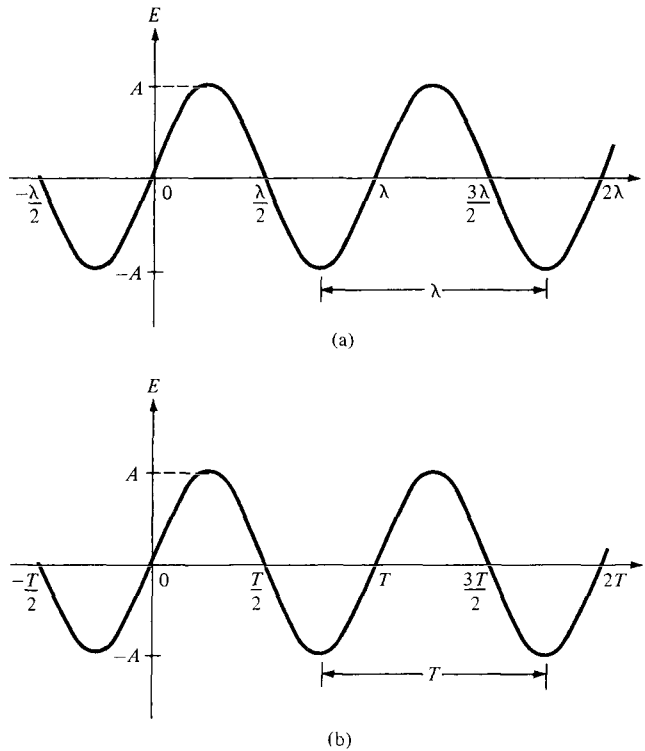


Figure 10.1 Plot of $E(z, t) = A \sin(\omega t - \beta z)$: (a) with constant t , (b) with constant z .

we expect from eqs. (10.6) and (10.7) that

$$\beta = \frac{2\pi}{\lambda} \quad (10.8)$$

Equation (10.8) shows that for every wavelength of distance traveled, a wave undergoes a phase change of 2π radians.

We will now show that the wave represented by eq. (10.5) is traveling with a velocity u in the $+z$ direction. To do this, we consider a fixed point P on the wave. We sketch eq. (10.5) at times $t = 0, T/4$, and $T/2$ as in Figure 10.2. From the figure, it is evident that as the wave advances with time, point P moves along $+z$ direction. Point P is a point of constant phase, therefore

$$\omega t - \beta z = \text{constant}$$

or

$$\frac{dz}{dt} = \frac{\omega}{\beta} = u \quad (10.9)$$

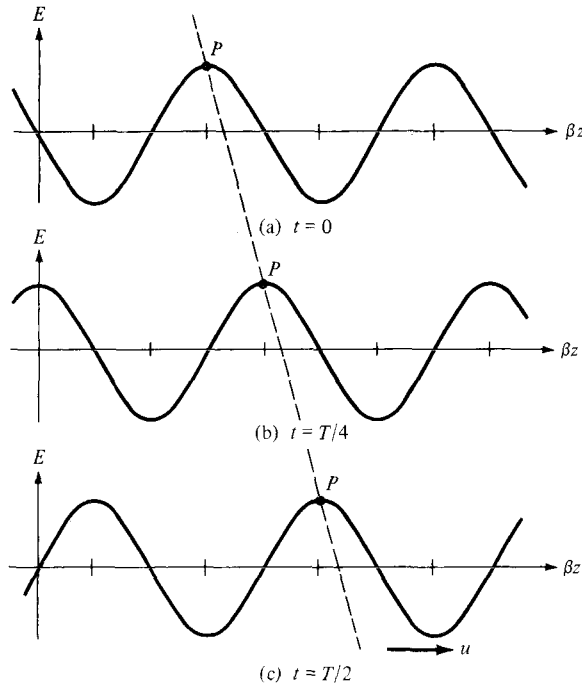


Figure 10.2 Plot of $E(z, t) = A \sin(\omega t - \beta z)$ at time (a) $t = 0$, (b) $t = T/4$, (c) $t = T/2$; P moves along $+z$ direction with velocity u .

which is the same as eq. (10.7b). Equation (10.9) shows that the wave travels with velocity u in the $+z$ direction. Similarly, it can be shown that the wave $B \sin(\omega t + \beta z)$ in eq. (10.4b) is traveling with velocity u in the $-z$ direction.

In summary, we note the following:

1. A wave is a function of both time and space.
2. Though time $t = 0$ is arbitrarily selected as a reference for the wave, a wave is without beginning or end.
3. A negative sign in $(\omega t \pm \beta z)$ is associated with a wave propagating in the $+z$ direction (forward traveling or positive-going wave) whereas a positive sign indicates that a wave is traveling in the $-z$ direction (backward traveling or negative-going wave).
4. Since $\sin(-\psi) = -\sin \psi = \sin(\psi \pm \pi)$, whereas $\cos(-\psi) = \cos \psi$,

$$\sin(\psi \pm \pi/2) = \pm \cos \psi \quad (10.10a)$$

$$\sin(\psi \pm \pi) = -\sin \psi \quad (10.10b)$$

$$\cos(\psi \pm \pi/2) = \mp \sin \psi \quad (10.10c)$$

$$\cos(\psi \pm \pi) = -\cos \psi \quad (10.10d)$$

where $\psi = \omega t \pm \beta z$. With eq. (10.10), any time-harmonic wave can be represented in the form of sine or cosine.

TABLE 10.1 Electromagnetic Spectrum

EM Phenomena	Examples of Uses	Approximate Frequency Range
Cosmic rays	Physics, astronomy	10^{14} GHz and above
Gamma rays	Cancer therapy	10^{10} – 10^{13} GHz
X-rays	X-ray examination	10^8 – 10^9 GHz
Ultraviolet radiation	Sterilization	10^6 – 10^8 GHz
Visible light	Human vision	10^5 – 10^6 GHz
Infrared radiation	Photography	10^3 – 10^4 GHz
Microwave waves	Radar, microwave relays, satellite communication	3–300 GHz
Radio waves	UHF television	470–806 MHz
	VHF television, FM radio	54–216 MHz
	Short-wave radio	3–26 MHz
	AM radio	535–1605 kHz

A large number of frequencies visualized in numerical order constitute a *spectrum*. Table 10.1 shows at what frequencies various types of energy in the EM spectrum occur. Frequencies usable for radio communication occur near the lower end of the EM spectrum. As frequency increases, the manifestation of EM energy becomes dangerous to human beings.¹ Microwave ovens, for example, can pose a hazard if not properly shielded. The practical difficulties of using EM energy for communication purposes also increase as frequency increases, until finally it can no longer be used. As communication methods improve, the limit to usable frequency has been pushed higher. Today communication satellites use frequencies near 14 GHz. This is still far below light frequencies, but in the enclosed environment of fiber optics, light itself can be used for radio communication.²

EXAMPLE 10.1

The electric field in free space is given by

$$\mathbf{E} = 50 \cos(10^8 t + \beta x) \mathbf{a}_y \text{ V/m}$$

- Find the direction of wave propagation.
- Calculate β and the time it takes to travel a distance of $\lambda/2$.
- Sketch the wave at $t = 0$, $T/4$, and $T/2$.

Solution:

- From the positive sign in $(\omega t + \beta x)$, we infer that the wave is propagating along $-\mathbf{a}_x$. This will be confirmed in part (c) of this example.

¹See March 1987 special issue of *IEEE Engineering in Medicine and Biology Magazine* on “Effects of EM Radiation.”

²See October 1980 issue of *IEEE Proceedings* on “Optical-Fiber Communications.”

(b) In free space, $u = c$.

$$\beta = \frac{\omega}{c} = \frac{10^8}{3 \times 10^8} = \frac{1}{3}$$

or

$$\beta = 0.3333 \text{ rad/m}$$

If T is the period of the wave, it takes T seconds to travel a distance λ at speed c . Hence to travel a distance of $\lambda/2$ will take

$$t_1 = \frac{T}{2} = \frac{1}{2} \frac{2\pi}{\omega} = \frac{\pi}{10^8} = 31.42 \text{ ns}$$

Alternatively, because the wave is traveling at the speed of light c ,

$$\frac{\lambda}{2} = ct_1 \quad \text{or} \quad t_1 = \frac{\lambda}{2c}$$

But

$$\lambda = \frac{2\pi}{\beta} = 6\pi$$

Hence,

$$t_1 = \frac{6\pi}{2(3 \times 10^8)} = 31.42 \text{ ns}$$

as obtained before.

$$(c) \text{ At } t = 0, \quad E_y = 50 \cos \beta x$$

$$\begin{aligned} \text{At } t = T/4, E_y &= 50 \cos \left(\omega \cdot \frac{2\pi}{4\omega} + \beta x \right) = 50 \cos (\beta x + \pi/2) \\ &= -50 \sin \beta x \end{aligned}$$

$$\begin{aligned} \text{At } t = T/2, E_y &= 50 \cos \left(\omega \cdot \frac{2\pi}{2\omega} + \beta x \right) = 50 \cos (\beta x + \pi) \\ &= -50 \cos \beta x \end{aligned}$$

E_y at $t = 0, T/4, T/2$ is plotted against x as shown in Figure 10.3. Notice that a point P (arbitrarily selected) on the wave moves along $-\mathbf{a}_x$ as t increases with time. This shows that the wave travels along $-\mathbf{a}_x$.

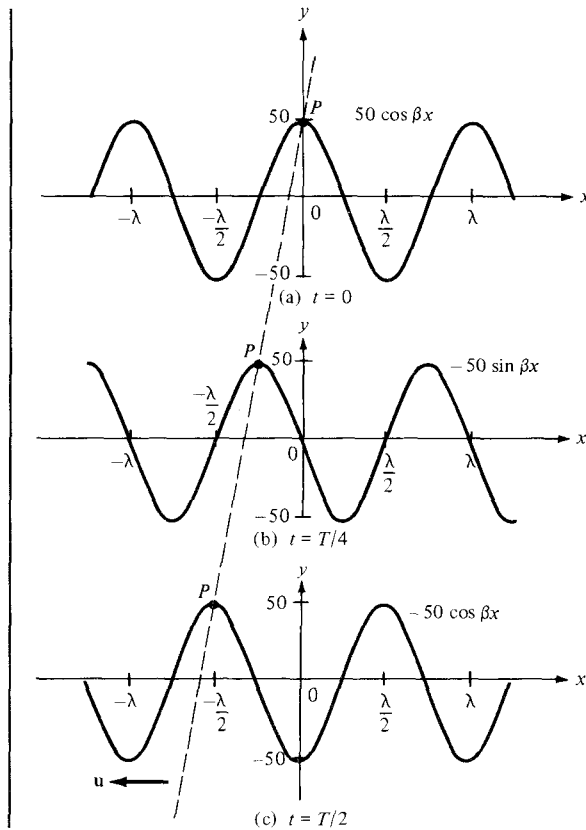


Figure 10.3 For Example 10.1; wave travels along $-\mathbf{a}_x$.

PRACTICE EXERCISE 10.1

In free space, $\mathbf{H} = 0.1 \cos(2 \times 10^8 t - kx) \mathbf{a}_y$ A/m. Calculate

- k , λ , and T
- The time t_1 it takes the wave to travel $\lambda/8$
- Sketch the wave at time t_1 .

Answer: (a) 0.667 rad/m, 9.425 m, 31.42 ns, (b) 3.927 ns, (c) see Figure 10.4.

10.3 WAVE PROPAGATION IN LOSSY DIELECTRICS

As mentioned in Section 10.1, wave propagation in lossy dielectrics is a general case from which wave propagation in other types of media can be derived as special cases. Therefore, this section is foundational to the next three sections.

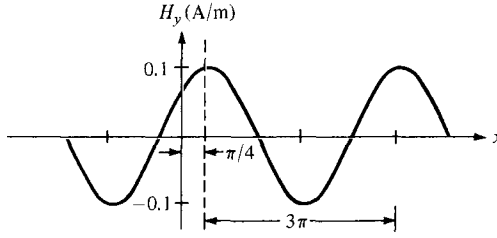


Figure 10.4 For Practice Exercise 10.1(c).

A lossy dielectric is a medium in which an EM wave loses power as it propagates due to poor conduction.

In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect or good dielectric) in which $\sigma = 0$.

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$). Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's equations (see Table 9.2) become

$$\nabla \cdot \mathbf{E}_s = 0 \quad (10.11)$$

$$\nabla \cdot \mathbf{H}_s = 0 \quad (10.12)$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \quad (10.13)$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\epsilon)\mathbf{E}_s \quad (10.14)$$

Taking the curl of both sides of eq. (10.13) gives

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu \nabla \times \mathbf{H}_s \quad (10.15)$$

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (10.16)$$

to the left-hand side of eq. (10.15) and invoking eqs. (10.11) and (10.14), we obtain

$$\nabla(\nabla \cdot \mathbf{E}_s) - \nabla^2 \mathbf{E}_s = -j\omega\mu(\sigma + j\omega\epsilon)\mathbf{E}_s$$

or

or

$$\boxed{\nabla^2 \mathbf{E}_s - \gamma^2 \mathbf{E}_s = 0} \quad (10.17)$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) \quad (10.18)$$

and γ is called the *propagation constant* (in per meter) of the medium. By a similar procedure, it can be shown that for the $\mathbf{H}$ field,

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0 \quad (10.19)$$

Equations (10.17) and (10.19) are known as homogeneous vector *Helmholtz's equations* or simply vector *wave equations*. In Cartesian coordinates, eq. (10.17), for example, is equivalent to three scalar wave equations, one for each component of $\mathbf{E}$ along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

Since γ in eqs. (10.17) to (10.19) is a complex quantity, we may let

$$\gamma = \alpha + j\beta \quad (10.20)$$

We obtain α and β from eqs. (10.18) and (10.20) by noting that

$$-\text{Re } \gamma^2 = \beta^2 - \alpha^2 = \omega^2 \mu \epsilon \quad (10.21)$$

and

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega \mu \sqrt{\sigma^2 + \omega^2 \epsilon^2} \quad (10.22)$$

From eqs. (10.21) and (10.22), we obtain

$$\alpha = \omega \sqrt{\frac{\mu \epsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega \epsilon} \right]^2} - 1 \right]} \quad (10.23)$$

$$\beta = \omega \sqrt{\frac{\mu \epsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega \epsilon} \right]^2} + 1 \right]} \quad (10.24)$$

Without loss of generality, if we assume that the wave propagates along $+\mathbf{a}_z$ and that $\mathbf{E}_s$ has only an x -component, then

$$\mathbf{E}_s = E_{xs}(z) \mathbf{a}_x \quad (10.25)$$

Substituting this into eq. (10.17) yields

$$(\nabla^2 - \gamma^2) E_{xs}(z) = 0 \quad (10.26)$$

Hence

$$\underbrace{\frac{\partial^2 E_{xs}(z)}{\partial x^2}}_0 + \underbrace{\frac{\partial^2 E_{xs}(z)}{\partial y^2}}_0 + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

or

$$\left[\frac{d^2}{dz^2} - \gamma^2 \right] E_{xs}(z) = 0 \quad (10.27)$$

This is a scalar wave equation, a linear homogeneous differential equation, with solution (see Case 2 in Example 6.5)

$$E_{xs}(z) = E_o e^{-\gamma z} + E'_o e^{\gamma z} \quad (10.28)$$

where E_o and E'_o are constants. The fact that the field must be finite at infinity requires that $E'_o = 0$. Alternatively, because $e^{\gamma z}$ denotes a wave traveling along $-\mathbf{a}_z$ whereas we assume wave propagation along $\mathbf{a}_z$, $E'_o = 0$. Whichever way we look at it, $E'_o = 0$. Inserting the time factor $e^{j\omega t}$ into eq. (10.28) and using eq. (10.20), we obtain

$$\mathbf{E}(z, t) = \text{Re} [E_{xs}(z) e^{j\omega t} \mathbf{a}_x] = \text{Re} (E_o e^{-\alpha z} e^{j(\omega t - \beta z)} \mathbf{a}_x)$$

or

$$\mathbf{E}(z, t) = E_o e^{-\alpha z} \cos(\omega t - \beta z) \mathbf{a}_x \quad (10.29)$$

A sketch of $|\mathbf{E}|$ at times $t = 0$ and $t = \Delta t$ is portrayed in Figure 10.5, where it is evident that $\mathbf{E}$ has only an x -component and it is traveling along the $+z$ -direction. Having obtained $\mathbf{E}(z, t)$, we obtain $\mathbf{H}(z, t)$ either by taking similar steps to solve eq. (10.19) or by using eq. (10.29) in conjunction with Maxwell's equations as we did in Example 9.8. We will eventually arrive at

$$\mathbf{H}(z, t) = \text{Re} (H_o e^{-\alpha z} e^{j(\omega t - \beta z)} \mathbf{a}_y) \quad (10.30)$$

where

$$H_o = \frac{E_o}{\eta} \quad (10.31)$$

and η is a complex quantity known as the *intrinsic impedance* (in ohms) of the medium. It can be shown by following the steps taken in Example 9.8 that

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta|/\angle\eta = |\eta|e^{j\theta_\eta} \quad (10.32)$$

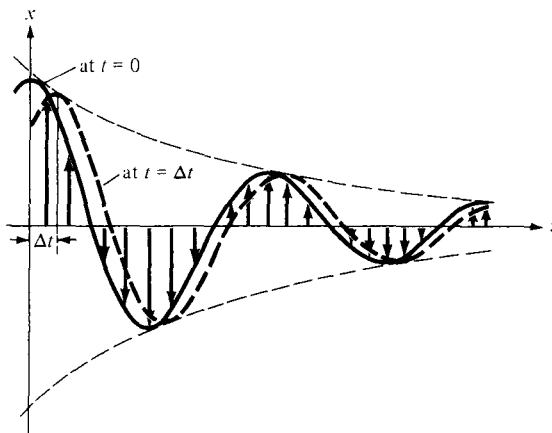


Figure 10.5 E -field with x -component traveling along $+z$ -direction at times $t = 0$ and $t = \Delta t$; arrows indicate instantaneous values of E .

with

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}}, \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} \quad (10.33)$$

where $0 \leq \theta_\eta \leq 45^\circ$. Substituting eqs. (10.31) and (10.32) into eq. (10.30) gives

$$\mathbf{H} = \text{Re} \left[\frac{E_o}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)} \mathbf{a}_y \right]$$

or

$$\mathbf{H} = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \mathbf{a}_y \quad (10.34)$$

Notice from eqs. (10.29) and (10.34) that as the wave propagates along $\mathbf{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, and hence α is known as the *attenuation constant* or *attenuation factor* of the medium. It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) or in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to e^{-1} of the original value whereas an increase of 1 neper indicates an increase by a factor of e . Hence, for voltages

$$1 \text{ Np} = 20 \log_{10} e = 8.686 \text{ dB} \quad (10.35)$$

From eq. (10.23), we notice that if $\sigma = 0$, as is the case for a lossless medium and free space, $\alpha = 0$ and the wave is not attenuated as it propagates. The quantity β is a measure of the phase shift per length and is called the *phase constant* or *wave number*. In terms of β , the wave velocity u and wavelength λ are, respectively, given by [see eqs. (10.7b) and (10.8)]

$$u = \frac{\omega}{\beta}, \quad \lambda = \frac{2\pi}{\beta} \quad (10.36)$$

We also notice from eqs. (10.29) and (10.34) that $\mathbf{E}$ and $\mathbf{H}$ are out of phase by θ_η at any instant of time due to the complex intrinsic impedance of the medium. Thus at any time, $\mathbf{E}$ leads $\mathbf{H}$ (or $\mathbf{H}$ lags $\mathbf{E}$) by θ_η . Finally, we notice that the ratio of the magnitude of the conduction current density $\mathbf{J}$ to that of the displacement current density $\mathbf{J}_d$ in a lossy medium is

$$\frac{|\mathbf{J}_s|}{|\mathbf{J}_{ds}|} = \frac{|\sigma \mathbf{E}_s|}{|j\omega\epsilon \mathbf{E}_s|} = \frac{\sigma}{\omega\epsilon} = \tan \theta$$

or

$$\tan \theta = \frac{\sigma}{\omega\epsilon} \quad (10.37)$$

where $\tan \theta$ is known as the *loss tangent* and θ is the *loss angle* of the medium as illustrated in Figure 10.6. Although a line of demarcation between good conductors and lossy dielectrics is not easy to make, $\tan \theta$ or θ may be used to determine how lossy a medium is. A medium is said to be a good (lossless or perfect) dielectric if $\tan \theta$ is very small ($\sigma \ll \omega\epsilon$) or a good conductor if $\tan \theta$ is very large ($\sigma \gg \omega\epsilon$). From the viewpoint of wave propagation, the characteristic behavior of a medium depends not only on its constitutive parameters σ , ϵ , and μ but also on the frequency of operation. A medium that is regarded as a good conductor at low frequencies may be a good dielectric at high frequencies. Note from eqs. (10.33) and (10.37) that

$$\theta = 2\theta_\eta \quad (10.38)$$

From eq. (10.14)

$$\begin{aligned} \nabla \times \mathbf{H}_s &= (\sigma + j\omega\epsilon)\mathbf{E}_s = j\omega\epsilon \left[1 - \frac{j\sigma}{\omega\epsilon} \right] \mathbf{E}_s \\ &= j\omega\epsilon_c \mathbf{E}_s \end{aligned} \quad (10.39)$$

where

$$\epsilon_c = \epsilon \left[1 - j \frac{\sigma}{\omega\epsilon} \right] \quad (10.40a)$$

or

$$\epsilon_c = \epsilon' - j\epsilon'' \quad (10.40b)$$

and $\epsilon' = \epsilon$, $\epsilon'' = \sigma/\omega$; ϵ_c is called the *complex permittivity* of the medium. We observe that the ratio of ϵ'' to ϵ' is the loss tangent of the medium; that is,

$$\tan \theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} \quad (10.41)$$

In subsequent sections, we will consider wave propagation in other types of media, which may be regarded as special cases of what we have considered here. Thus we will simply deduce the governing formulas from those obtained for the general case treated in this section. The student is advised not just to memorize the formulas but to observe how they are easily obtained from the formulas for the general case.

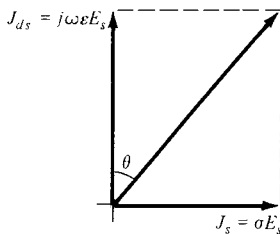


Figure 10.6 Loss angle of a lossy medium.

10.4 PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\epsilon$. It is a special case of that in Section 10.3 except that

$$\sigma \simeq 0, \quad \epsilon = \epsilon_0 \epsilon_r, \quad \mu = \mu_0 \mu_r \quad (10.42)$$

Substituting these into eqs. (10.23) and (10.24) gives

$$\alpha = 0, \quad \beta = \omega \sqrt{\mu\epsilon} \quad (10.43a)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}, \quad \lambda = \frac{2\pi}{\beta} \quad (10.43b)$$

Also

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \angle 0^\circ \quad (10.44)$$

and thus **E** and **H** are in time phase with each other.

10.5 PLANE WAVES IN FREE SPACE

This is a special case of what we considered in Section 10.3. In this case,

$$\sigma = 0, \quad \epsilon = \epsilon_0, \quad \mu = \mu_0 \quad (10.45)$$

This may also be regarded as a special case of Section 10.4. Thus we simply replace ϵ by ϵ_0 and μ by μ_0 in eq. (10.43) or we substitute eq. (10.45) directly into eqs. (10.23) and (10.24). Either way, we obtain

$$\alpha = 0, \quad \beta = \omega \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} \quad (10.46a)$$

$$u = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c, \quad \lambda = \frac{2\pi}{\beta} \quad (10.46b)$$

where $c \simeq 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM wave travels in free space at the speed of light is significant. It shows that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

By substituting the constitutive parameters in eq. (10.45) into eq. (10.33), $\theta_\eta = 0$ and $\eta = \eta_0$, where η_0 is called the *intrinsic impedance of free space* and is given by

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \approx 377 \Omega \quad (10.47)$$

$$\mathbf{E} = E_0 \cos(\omega t - \beta z) \mathbf{a}_x \quad (10.48a)$$

then

$$\mathbf{H} = H_0 \cos(\omega t - \beta z) \mathbf{a}_y = \frac{E_0}{\eta_0} \cos(\omega t - \beta z) \mathbf{a}_y \quad (10.48b)$$

The plots of $\mathbf{E}$ and $\mathbf{H}$ are shown in Figure 10.7(a). In general, if $\mathbf{a}_E$, $\mathbf{a}_H$, and $\mathbf{a}_k$ are unit vectors along the $\mathbf{E}$ field, the $\mathbf{H}$ field, and the direction of wave propagation; it can be shown that (see Problem 10.14)

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$$

or

$$\mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

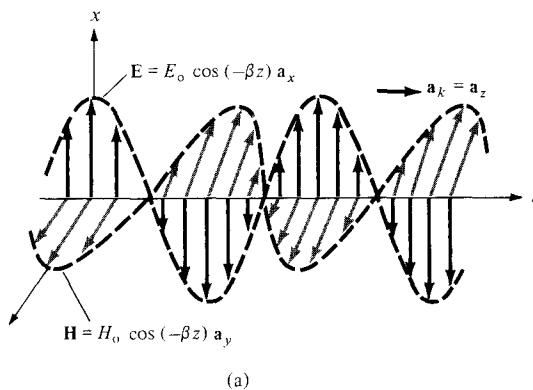
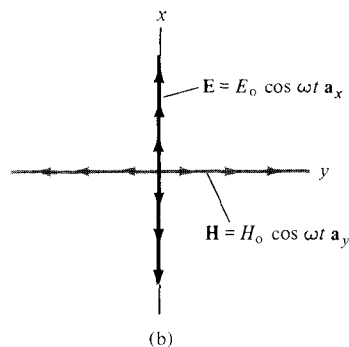


Figure 10.7 (a) Plot of $\mathbf{E}$ and $\mathbf{H}$ as functions of z at $t = 0$; (b) plot of $\mathbf{E}$ and $\mathbf{H}$ at $z = 0$. The arrows indicate instantaneous values.



or

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k \quad (10.49)$$

Both $\mathbf{E}$ and $\mathbf{H}$ fields (or EM waves) are everywhere normal to the direction of wave propagation, $\mathbf{a}_k$. That means, the fields lie in a plane that is transverse or orthogonal to the direction of wave propagation. They form an EM wave that has no electric or magnetic field components along the direction of propagation; such a wave is called a *transverse electromagnetic* (TEM) wave. Each of $\mathbf{E}$ and $\mathbf{H}$ is called a *uniform plane wave* because $\mathbf{E}$ (or $\mathbf{H}$) has the same magnitude throughout any transverse plane, defined by $z = \text{constant}$. The direction in which the electric field points is the *polarization* of a TEM wave.³ The wave in eq. (10.29), for example, is polarized in the x -direction. This should be observed in Figure 10.7(b), where an illustration of uniform plane waves is given. A uniform plane wave cannot exist physically because it stretches to infinity and would represent an infinite energy. However, such waves are characteristically simple but fundamentally important. They serve as approximations to practical waves, such as from a radio antenna, at distances sufficiently far from radiating sources. Although our discussion after eq. (10.48) deals with free space, it also applies for any other isotropic medium.

10.6 PLANE WAVES IN GOOD CONDUCTORS

This is another special case of that considered in Section 10.3. A perfect, or good conductor, is one in which $\sigma \gg \omega\epsilon$ so that $\sigma/\omega\epsilon \rightarrow \infty$; that is,

$$\sigma \simeq \infty, \quad \epsilon = \epsilon_0, \quad \mu = \mu_0\mu_r \quad (10.50)$$

Hence, eqs. (10.23) and (10.24) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \quad (10.51a)$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \quad \lambda = \frac{2\pi}{\beta} \quad (10.51b)$$

Also,

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \quad (10.52)$$

and thus $\mathbf{E}$ leads $\mathbf{H}$ by 45° . If

$$\mathbf{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \mathbf{a}_x \quad (10.53a)$$

³Some texts define polarization differently.

then

$$\mathbf{H} = \frac{E_o}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ) \mathbf{a}_y \quad (10.53b)$$

Therefore, as $\mathbf{E}$ (or $\mathbf{H}$) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ , shown in Figure 10.8, through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called *skin depth* or *penetration depth* of the medium; that is,

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

or

$$\delta = \frac{1}{\alpha} \quad (10.54a)$$

The **skin depth** is a measure of the depth to which an **EM** wave can penetrate the medium.

Equation (10.54a) is generally valid for any material medium. For good conductors, eqs. (10.51a) and (10.54a) give

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad (10.54b)$$

The illustration in Figure 10.8 for a good conductor is exaggerated. However, for a partially conducting medium, the skin depth can be considerably large. Note from eqs. (10.51a), (10.52), and (10.54b) that for a good conductor.

$$\eta = \frac{1}{\sigma\delta} \sqrt{2} e^{j\pi/4} = \frac{1+j}{\sigma\delta} \quad (10.55)$$

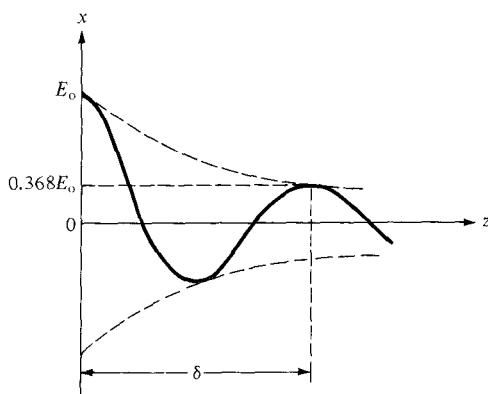


Figure 10.8 Illustration of skin depth.

TABLE 10.2 Skin Depth in Copper*

Frequency (Hz)	10	60	100	500	10 ⁴	10 ⁸	10 ¹⁰
Skin depth (mm)	20.8	8.6	6.6	2.99	0.66	6.6×10^{-3}	6.6×10^{-4}

*For copper, $\sigma = 5.8 \times 10^7$ mhos/m, $\mu = \mu_0$, $\delta = 66.1/\sqrt{f}$ (in mm).

Also for good conductors, eq. (10.53a) can be written as

$$\mathbf{E} = E_0 e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right) \mathbf{a}_x$$

showing that δ measures the exponential damping of the wave as it travels through the conductor. The skin depth in copper at various frequencies is shown in Table 10.2. From the table, we notice that the skin depth decreases with increase in frequency. Thus, $\mathbf{E}$ and $\mathbf{H}$ can hardly propagate through good conductors.

The phenomenon whereby field intensity in a conductor rapidly decreases is known as *skin effect*. The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface. For a wire of radius a , for example, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness δ as shown in Figure 10.9. Skin effect appears in different guises in such problems as attenuation in waveguides, effective or ac resistance of transmission lines, and electromagnetic shielding. It is used to advantage in many applications. For example, because the skin depth in silver is very small, the difference in performance between a pure silver component and a silver-plated brass component is negligible, so silver plating is often used to reduce material cost of waveguide components. For the same reason, hollow tubular conductors are used instead of solid conductors in outdoor television antennas. Effective electromagnetic shielding of electrical devices can be provided by conductive enclosures a few skin depths in thickness.

The skin depth is useful in calculating the *ac resistance* due to skin effect. The resistance in eq. (5.16) is called the *dc resistance*, that is,

$$R_{dc} = \frac{\ell}{\sigma S} \quad (5.16)$$

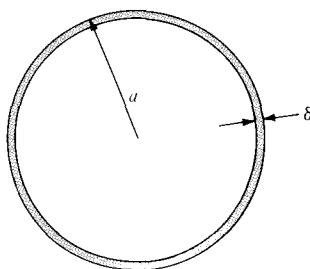


Figure 10.9 Skin depth at high frequencies, $\delta \ll a$.

We define the *surface or skin resistance* R_s (in Ω/m^2) as the real part of the η for a good conductor. Thus from eq. (10.55)

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}} \quad (10.56)$$

This is the resistance of a unit width and unit length of the conductor. It is equivalent to the dc resistance for a unit length of the conductor having cross-sectional area $1 \times \delta$. Thus for a given width w and length ℓ , the ac resistance is calculated using the familiar dc resistance relation of eq. (5.16) and assuming a uniform current flow in the conductor of thickness δ , that is,

$$R_{ac} = \frac{\ell}{\sigma\delta w} = \frac{R_s \ell}{w} \quad (10.57)$$

where $S \approx \delta w$. For a conductor wire of radius a (see Figure 10.9), $w = 2\pi a$, so

$$\frac{R_{ac}}{R_{dc}} = \frac{\frac{\ell}{\sigma 2\pi a \delta}}{\frac{\ell}{\sigma \pi a^2}} = \frac{a}{2\delta}$$

Since $\delta \ll a$ at high frequencies, this shows that R_{ac} is far greater than R_{dc} . In general, the ratio of the ac to the dc resistance starts at 1.0 for dc and very low frequencies and increases as the frequency increases. Also, although the bulk of the current is nonuniformly distributed over a thickness of 5δ of the conductor, the power loss is the same as though it were uniformly distributed over a thickness of δ and zero elsewhere. This is one more reason why δ is referred to as the skin depth.

EXAMPLE 10.2

A lossy dielectric has an intrinsic impedance of $200 \angle 30^\circ \Omega$ at a particular frequency. If, at that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$\mathbf{H} = 10 e^{-\alpha x} \cos\left(\omega t - \frac{1}{2}x\right) \mathbf{a}_y \text{ A/m}$$

find $\mathbf{E}$ and α . Determine the skin depth and wave polarization.

Solution:

The given wave travels along $\mathbf{a}_x$ so that $\mathbf{a}_k = \mathbf{a}_x$; $\mathbf{a}_H = \mathbf{a}_y$, so

$$-\mathbf{a}_E = \mathbf{a}_k \times \mathbf{a}_H = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

or

$$\mathbf{a}_E = -\mathbf{a}_z$$

Also $H_o = 10$, so

$$\frac{E_o}{H_o} = \eta = 200 \angle 30^\circ = 200 e^{j\pi/6} \rightarrow E_o = 2000 e^{j\pi/6}$$

Except for the amplitude and phase difference, $\mathbf{E}$ and $\mathbf{H}$ always have the same form. Hence

$$\mathbf{E} = \text{Re} (2000 e^{j\pi/6} e^{-\gamma x} e^{j\omega t} \mathbf{a}_E)$$

or

$$\mathbf{E} = -2e^{-\alpha x} \cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right) \mathbf{a}_z \text{ kV/m}$$

Knowing that $\beta = 1/2$, we need to determine α . Since

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon} \right]^2} - 1 \right]}$$

and

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon} \right]^2} + 1 \right]}$$

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon} \right]^2} - 1}{\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon} \right]^2} + 1} \right]^{1/2}$$

But $\frac{\sigma}{\omega\epsilon} = \tan 2\theta_\eta = \tan 60^\circ = \sqrt{3}$. Hence,

$$\frac{\alpha}{\beta} = \left[\frac{2 - 1}{2 + 1} \right]^{1/2} = \frac{1}{\sqrt{3}}$$

or

$$\alpha = \frac{\beta}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887 \text{ Np/m}$$

and

$$\delta = \frac{1}{\alpha} = 2\sqrt{3} = 3.464 \text{ m}$$

The wave has an E_z component; hence it is polarized along the z -direction.

PRACTICE EXERCISE 10.2

A plane wave propagating through a medium with $\epsilon_r = 8$, $\mu_r = 2$ has $\mathbf{E} = 0.5 e^{-z/3} \sin(10^8 t - \beta z) \mathbf{a}_x$ V/m. Determine

- (a) β
- (b) The loss tangent
- (c) Wave impedance
- (d) Wave velocity
- (e) $\mathbf{H}$ field

Answer: (a) 1.374 rad/m, (b) 0.5154, (c) 177.72 $\angle 13.63^\circ \Omega$, (d) 7.278×10^7 m/s, (e) $2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63^\circ) \mathbf{a}_y$ mA/m.

EXAMPLE 10.3

In a lossless medium for which $\eta = 60\pi$, $\mu_r = 1$, and $\mathbf{H} = -0.1 \cos(\omega t - z) \mathbf{a}_x + 0.5 \sin(\omega t - z) \mathbf{a}_y$ A/m, calculate ϵ_r , ω , and $\mathbf{E}$.

Solution:

In this case, $\sigma = 0$, $\alpha = 0$, and $\beta = 1$, so

$$\eta = \sqrt{\mu/\epsilon} = \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{\frac{\mu_r}{\epsilon_r}} = \frac{120\pi}{\sqrt{\epsilon_r}}$$

or

$$\sqrt{\epsilon_r} = \frac{120\pi}{\eta} = \frac{120\pi}{60\pi} = 2 \quad \rightarrow \quad \epsilon_r = 4$$

$$\beta = \omega \sqrt{\mu\epsilon} = \omega \sqrt{\mu_0\epsilon_0} \sqrt{\mu_r\epsilon_r} = \frac{\omega}{c} \sqrt{4} = \frac{2\omega}{c}$$

or

$$\omega = \frac{\beta c}{2} = \frac{1(3 \times 10^8)}{2} = 1.5 \times 10^8 \text{ rad/s}$$

From the given $\mathbf{H}$ field, $\mathbf{E}$ can be calculated in two ways: using the techniques (based on Maxwell's equations) developed in this chapter or directly using Maxwell's equations as in the last chapter.

Method 1: To use the techniques developed in this chapter, we let

$$\mathbf{E} = \mathbf{H}_1 + \mathbf{H}_2$$

where $\mathbf{H}_1 = -0.1 \cos(\omega t - z) \mathbf{a}_x$ and $\mathbf{H}_2 = 0.5 \sin(\omega t - z) \mathbf{a}_y$ and the corresponding electric field

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

where $\mathbf{E}_1 = E_{10} \cos(\omega t - z) \mathbf{a}_{E_1}$ and $\mathbf{E}_2 = E_{20} \sin(\omega t - z) \mathbf{a}_{E_2}$. Notice that although $\mathbf{H}$ has components along $\mathbf{a}_x$ and $\mathbf{a}_y$, it has no component along the direction of propagation; it is therefore a TEM wave.

For $\mathbf{E}_1$,

$$\mathbf{a}_{E_1} = -(\mathbf{a}_k \times \mathbf{a}_{H_1}) = -(\mathbf{a}_z \times -\mathbf{a}_x) = \mathbf{a}_y$$

$$E_{10} = \eta H_{10} = 60\pi (0.1) = 6\pi$$

Hence

$$\mathbf{E}_1 = 6\pi \cos(\omega t - z) \mathbf{a}_y$$

For $\mathbf{E}_2$,

$$\mathbf{a}_{E_2} = -(\mathbf{a}_k \times \mathbf{a}_{H_2}) = -(\mathbf{a}_z \times \mathbf{a}_y) = \mathbf{a}_x$$

$$E_{20} = \eta H_{20} = 60\pi (0.5) = 30\pi$$

Hence

$$\mathbf{E}_2 = 30\pi \sin(\omega t - z) \mathbf{a}_x$$

Adding $\mathbf{E}_1$ and $\mathbf{E}_2$ gives $\mathbf{E}$; that is,

$$\mathbf{E} = 94.25 \sin(1.5 \times 10^8 t - z) \mathbf{a}_x + 18.85 \cos(1.5 \times 10^8 t - z) \mathbf{a}_y \text{ V/m}$$

Method 2: We may apply Maxwell's equations directly.

$$\nabla \times \mathbf{H} = \underbrace{\sigma \mathbf{E}}_0 + \epsilon \frac{\partial \mathbf{E}}{\partial t} \rightarrow \mathbf{E} = \frac{1}{\epsilon} \int \nabla \times \mathbf{H} dt$$

because $\sigma = 0$. But

$$\begin{aligned} \nabla \times \mathbf{H} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(z) & H_y(z) & 0 \end{vmatrix} = -\frac{\partial H_y}{\partial z} \mathbf{a}_x + \frac{\partial H_x}{\partial z} \mathbf{a}_y \\ &= H_{20} \cos(\omega t - z) \mathbf{a}_x + H_{10} \sin(\omega t - z) \mathbf{a}_y \end{aligned}$$

where $H_{10} = -0.1$ and $H_{20} = 0.5$. Hence

$$\begin{aligned} \mathbf{E} &= \frac{1}{\epsilon} \int \nabla \times \mathbf{H} dt = \frac{H_{20}}{\epsilon \omega} \sin(\omega t - z) \mathbf{a}_x - \frac{H_{10}}{\epsilon \omega} \cos(\omega t - z) \mathbf{a}_y \\ &= 94.25 \sin(\omega t - z) \mathbf{a}_x + 18.85 \cos(\omega t - z) \mathbf{a}_y \text{ V/m} \end{aligned}$$

as expected.

PRACTICE EXERCISE 10.3

A plane wave in a nonmagnetic medium has $\mathbf{E} = 50 \sin(10^8 t + 2z) \mathbf{a}_y$ V/m. Find

- (a) The direction of wave propagation
- (b) λ , f , and ϵ_r
- (c) $\mathbf{H}$

Answer: (a) along $-z$ direction, (b) 3.142 m, 15.92 MHz, 36, (c) $0.7958 \sin(10^8 t + 2z) \mathbf{a}_x$ A/m.

EXAMPLE 10.4

A uniform plane wave propagating in a medium has

$$\mathbf{E} = 2e^{-\alpha z} \sin(10^8 t - \beta z) \mathbf{a}_y \text{ V/m.}$$

If the medium is characterized by $\epsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ mhos/m, find α , β , and $\mathbf{H}$.

Solution:

We need to determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega\epsilon} = \frac{3}{10^8 \times 1 \times \frac{10^{-9}}{36\pi}} = 3393 \gg 1$$

showing that the medium may be regarded as a good conductor at the frequency of operation. Hence,

$$\begin{aligned} \alpha = \beta &= \sqrt{\frac{\mu\omega\sigma}{2}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)(3)}{2} \right]^{1/2} \\ &= 61.4 \\ \alpha &= 61.4 \text{ Np/m}, \quad \beta = 61.4 \text{ rad/m} \end{aligned}$$

Also

$$\begin{aligned} |\eta| &= \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)}{3} \right]^{1/2} \\ &= \sqrt{\frac{800\pi}{3}} \end{aligned}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = 3393 \quad \rightarrow \quad \theta_\eta = 45^\circ = \pi/4$$

Hence

$$\mathbf{H} = H_0 e^{-\alpha z} \sin\left(\omega t - \beta z - \frac{\pi}{4}\right) \mathbf{a}_H$$

where

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

and

$$H_o = \frac{E_o}{|\eta|} = 2 \sqrt{\frac{3}{800\pi}} = 69.1 \times 10^{-3}$$

Thus

$$\mathbf{H} = -69.1 e^{-61.4z} \sin\left(10^8 t - 61.42z - \frac{\pi}{4}\right) \mathbf{a}_x \text{ mA/m}$$

PRACTICE EXERCISE 10.4

A plane wave traveling in the $+y$ -direction in a lossy medium ($\epsilon_r = 4$, $\mu_r = 1$, $\sigma = 10^{-2}$ mhos/m) has $\mathbf{E} = 30 \cos(10^9 \pi t + \pi/4) \mathbf{a}_z$ V/m at $y = 0$. Find

- (a) $\mathbf{E}$ at $y = 1$ m, $t = 2$ ns
- (b) The distance traveled by the wave to have a phase shift of 10°
- (c) The distance traveled by the wave to have its amplitude reduced by 40%
- (d) $\mathbf{H}$ at $y = 2$ m, $t = 2$ ns

Answer: (a) $2.787 \mathbf{a}_z$ V/m, (b) 8.325 mm, (c) 542 mm, (d) $-4.71 \mathbf{a}_x$ mA/m

EXAMPLE 10.5

A plane wave $\mathbf{E} = E_o \cos(\omega t - \beta z) \mathbf{a}_x$ is incident on a good conductor at $z = 0$. Find the current density in the conductor.

Solution:

Since the current density $\mathbf{J} = \sigma \mathbf{E}$, we expect $\mathbf{J}$ to satisfy the wave equation in eq. (10.17), that is,

$$\nabla^2 \mathbf{J}_s - \gamma^2 \mathbf{J}_s = 0$$

Also the incident $\mathbf{E}$ has only an x -component and varies with z . Hence $\mathbf{J} = J_x(z, t) \mathbf{a}_x$ and

$$\frac{d^2}{dz^2} J_{sx} - \gamma^2 J_{sx} = 0$$

which is an ordinary differential equation with solution (see Case 2 of Example 6.5)

$$J_{sx} = A e^{-\gamma z} + B e^{+\gamma z}$$

The constant B must be zero because J_{sx} is finite as $z \rightarrow \infty$. But in a good conductor, $\sigma \gg \omega\epsilon$ so that $\alpha = \beta = 1/\delta$. Hence

$$\gamma = \alpha + j\beta = \alpha(1 + j) = \frac{(1 + j)}{\delta}$$

and

$$J_{sx} = Ae^{-z(1+j)/\delta}$$

or

$$J_{sx} = J_{sx}(0) e^{-z(1+j)/\delta}$$

where $J_{sx}(0)$ is the current density on the conductor surface.

PRACTICE EXERCISE 10.5

Due to the current density of Example 10.5, find the magnitude of the total current through a strip of the conductor of infinite depth along z and width w along y .

Answer: $\frac{J_{sx}(0)w\delta}{\sqrt{2}}$

EXAMPLE 10.6

For the copper coaxial cable of Figure 7.12, let $a = 2$ mm, $b = 6$ mm, and $t = 1$ mm. Calculate the resistance of 2 m length of the cable at dc and at 100 MHz.

Solution:

Let

$$R = R_o + R_i$$

where R_o and R_i are the resistances of the inner and outer conductors.

At dc,

$$\begin{aligned} R_i &= \frac{\ell}{\sigma S} = \frac{\ell}{\sigma \pi a^2} = \frac{2}{5.8 \times 10^7 \pi [2 \times 10^{-3}]^2} = 2.744 \text{ m}\Omega \\ R_o &= \frac{\ell}{\sigma S} = \frac{\ell}{\sigma \pi [(b+t)^2 - b^2]} = \frac{\ell}{\sigma \pi [t^2 + 2bt]} \\ &= \frac{2}{5.8 \times 10^7 \pi [1 + 12] \times 10^{-6}} \\ &= 0.8429 \text{ m}\Omega \end{aligned}$$

$$\text{Hence } R_{dc} = 2.744 + 0.8429 = 3.587 \text{ m}\Omega$$

At $f = 100$ MHz,

$$\begin{aligned} R_i &= \frac{R_s \ell}{w} = \frac{\ell}{\sigma \delta 2\pi a} = \frac{\ell}{2\pi a} \sqrt{\frac{\pi f \mu}{\sigma}} \\ &= \frac{2}{2\pi \times 2 \times 10^{-3}} \sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} \\ &= 0.41 \, \Omega \end{aligned}$$

Since $\delta = 6.6 \, \mu\text{m} \ll t = 1$ mm, $w = 2\pi b$ for the outer conductor. Hence,

$$\begin{aligned} R_o &= \frac{R_s \ell}{w} = \frac{\ell}{2\pi b} \sqrt{\frac{\pi f \mu}{\sigma}} \\ &= \frac{2}{2\pi \times 6 \times 10^{-3}} \sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} \\ &= 0.1384 \, \Omega \end{aligned}$$

Hence,

$$R_{ac} = 0.41 + 0.1384 = 0.5484 \, \Omega$$

which is about 150 times greater than R_{dc} . Thus, for the same effective current i , the ohmic loss ($i^2 R$) of the cable at 100 MHz is far greater than the dc power loss by a factor of 150.

PRACTICE EXERCISE 10.6

For an aluminum wire having a diameter 2.6 mm, calculate the ratio of ac to dc resistance at

- (a) 10 MHz
- (b) 2 GHz

Answer: (a) 24.16, (b) 341.7.

0.7 POWER AND THE POYNTING VECTOR

As mentioned before, energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (10.58a)$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad (10.58b)$$

Dotting both sides of eq. (10.58b) with $\mathbf{E}$ gives

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad (10.59)$$

But for any vector fields $\mathbf{A}$ and $\mathbf{B}$ (see Appendix A.10)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

Applying this vector identity to eq. (10.59) (letting $\mathbf{A} = \mathbf{H}$ and $\mathbf{B} = \mathbf{E}$) gives

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) + \nabla \cdot (\mathbf{H} \times \mathbf{E}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad (10.60)$$

From eq. (10.58a),

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{H} \cdot \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{\mu}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H}) \quad (10.61)$$

and thus eq. (10.60) becomes

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the volume integral of both sides,

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv \quad (10.62)$$

Applying the divergence theorem to the left-hand side gives

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv \quad (10.63)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{Total power} & \text{Rate of decrease in} & \text{Ohmic power} \\ \text{leaving the volume} & \text{energy stored in electric} & \text{dissipated} \\ & \text{and magnetic fields} & \end{array} \quad (10.64)$$

Equation (10.63) is referred to as *Poynting's theorem*.⁴ The various terms in the equation are identified using energy-conservation arguments for EM fields. The first term on the right-hand side of eq. (10.63) is interpreted as the rate of decrease in energy stored in the electric and magnetic fields. The second term is the power dissipated due to the fact that the medium is conducting ($\sigma \neq 0$). The quantity $\mathbf{E} \times \mathbf{H}$ on the left-hand side of eq. (10.63) is known as the *Poynting vector* $\mathcal{P}$ in watts per square meter (W/m^2); that is,

$$\mathcal{P} = \mathbf{E} \times \mathbf{H} \quad (10.65)$$

⁴After J. H. Poynting, "On the transfer of energy in the electromagnetic field," *Phil. Trans.*, vol. 174, 1883, p. 343.

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the conduction losses.

The theorem is illustrated in Figure 10.10.

It should be noted that $\mathcal{P}$ is normal to both $\mathbf{E}$ and $\mathbf{H}$ and is therefore along the direction of wave propagation $\mathbf{a}_k$ for uniform plane waves. Thus

$$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H \quad (10.49)$$

The fact that $\mathcal{P}$ points along $\mathbf{a}_k$ causes $\mathcal{P}$ to be regarded derisively as a "pointing" vector.

Again, if we assume that

$$\mathbf{E}(z, t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \mathbf{a}_x$$

then

$$\mathbf{H}(z, t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \mathbf{a}_y$$

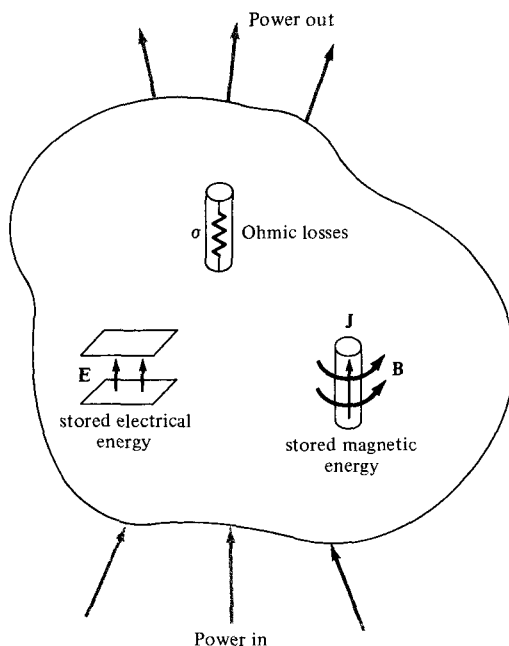


Figure 10.10 Illustration of power balance for EM fields.

and

$$\begin{aligned}\mathcal{P}(z, t) &= \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) \mathbf{a}_z \\ &= \frac{E_o^2}{2|\eta|} e^{-2\alpha z} [\cos \theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)] \mathbf{a}_z\end{aligned}\quad (10.66)$$

since $\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$. To determine the time-average Poynting vector $\mathcal{P}_{\text{ave}}(z)$ (in W/m²), which is of more practical value than the instantaneous Poynting vector $\mathcal{P}(z, t)$, we integrate eq. (10.66) over the period $T = 2\pi/\omega$; that is,

$$\mathcal{P}_{\text{ave}}(z) = \frac{1}{T} \int_0^T \mathcal{P}(z, t) dt \quad (10.67)$$

It can be shown (see Prob. 10.28) that this is equivalent to

$$\mathcal{P}_{\text{ave}}(z) = \frac{1}{2} \text{Re}(\mathbf{E}_s \times \mathbf{H}_s^*) \quad (10.68)$$

By substituting eq. (10.66) into eq. (10.67), we obtain

$$\mathcal{P}_{\text{ave}}(z) = \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos \theta_\eta \mathbf{a}_z \quad (10.69)$$

The total time-average power crossing a given surface S is given by

$$P_{\text{ave}} = \int_S \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} \quad (10.70)$$

We should note the difference between $\mathcal{P}$, $\mathcal{P}_{\text{ave}}$, and P_{ave} . $\mathcal{P}(x, y, z, t)$ is the Poynting vector in watts/meter and is time varying. $\mathcal{P}_{\text{ave}}(x, y, z)$ also in watts/meter is the time average of the Poynting vector $\mathcal{P}$; it is a vector but is time invariant. P_{ave} is a total time-average power through a surface in watts; it is a scalar.

EXAMPLE 10.7

In a nonmagnetic medium

$$\mathbf{E} = 4 \sin(2\pi \times 10^7 t - 0.8x) \mathbf{a}_z \text{ V/m}$$

Find

- (a) ϵ_r, η
- (b) The time-average power carried by the wave
- (c) The total power crossing 100 cm^2 of plane $2x + y = 5$

Solution:

- (a) Since $\alpha = 0$ and $\beta \neq \omega/c$, the medium is not free space but a lossless medium.

$$\beta = 0.8, \quad \omega = 2\pi \times 10^7, \quad \mu = \mu_0 \text{ (nonmagnetic)}, \quad \epsilon = \epsilon_0 \epsilon_r$$

Hence

$$\beta = \omega \sqrt{\mu \epsilon} = \omega \sqrt{\mu_0 \epsilon_0 \epsilon_r} = \frac{\omega}{c} \sqrt{\epsilon_r}$$

or

$$\sqrt{\epsilon_r} = \frac{\beta c}{\omega} = \frac{0.8(3 \times 10^8)}{2\pi \times 10^7} = \frac{12}{\pi}$$

$$\epsilon_r = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_r}} = \frac{120\pi}{\sqrt{\epsilon_r}} = 120\pi \cdot \frac{\pi}{12} = 10\pi^2$$

$$= 98.7 \, \Omega$$

$$(b) \quad \mathcal{P} = \mathbf{E} \times \mathbf{H} = \frac{E_o^2}{\eta} \sin^2(\omega t - \beta x) \mathbf{a}_x$$

$$\mathcal{P}_{\text{ave}} = \frac{1}{T} \int_0^T \mathcal{P} \, dt = \frac{E_o^2}{2\eta} \mathbf{a}_x = \frac{16}{2 \times 10\pi^2} \mathbf{a}_x$$

$$= 81 \mathbf{a}_x \text{ mW/m}^2$$

- (c) On plane $2x + y = 5$ (see Example 3.5 or 8.5),

$$\mathbf{a}_n = \frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}$$

Hence the total power is

$$P_{\text{ave}} = \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} = \mathcal{P}_{\text{ave}} \cdot S \mathbf{a}_n$$

$$= (81 \times 10^{-3} \mathbf{a}_x) \cdot (100 \times 10^{-4}) \left[\frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}} \right]$$

$$= \frac{162 \times 10^{-5}}{\sqrt{5}} = 724.5 \, \mu\text{W}$$

PRACTICE EXERCISE 10.7

In free space, $\mathbf{H} = 0.2 \cos(\omega t - \beta x) \mathbf{a}_z$ A/m. Find the total power passing through:

- (a) A square plate of side 10 cm on plane $x + z = 1$
- (b) A circular disc of radius 5 cm on plane $x = 1$.

Answer: (a) 0, (b) 59.22 mW.

10.8 REFLECTION OF A PLANE WAVE AT NORMAL INCIDENCE

So far, we have considered uniform plane waves traveling in unbounded, homogeneous media. When a plane wave from one medium meets a different medium, it is partly reflected and partly transmitted. The proportion of the incident wave that is reflected or transmitted depends on the constitutive parameters (ϵ , μ , σ) of the two media involved. Here we will assume that the incident wave plane is normal to the boundary between the media; oblique incidence of plane waves will be covered in the next section after we understand the simpler case of normal incidence.

Suppose that a plane wave propagating along the $+z$ -direction is incident normally on the boundary $z = 0$ between medium 1 ($z < 0$) characterized by σ_1 , ϵ_1 , μ_1 and medium 2 ($z > 0$) characterized by σ_2 , ϵ_2 , μ_2 , as shown in Figure 10.11. In the figure, subscripts i , r , and t denote incident, reflected, and transmitted waves, respectively. The incident, reflected, and transmitted waves shown in Figure 10.11 are obtained as follows:

Incident Wave:

$(\mathbf{E}_i, \mathbf{H}_i)$ is traveling along $+\mathbf{a}_z$ in medium 1. If we suppress the time factor $e^{j\omega t}$ and assume that

$$\mathbf{E}_{is}(z) = E_{io}e^{-\gamma_1 z} \mathbf{a}_x \quad (10.71)$$

then

$$\mathbf{H}_{is}(z) = H_{io}e^{-\gamma_1 z} \mathbf{a}_y = \frac{E_{io}}{\eta_1} e^{-\gamma_1 z} \mathbf{a}_y \quad (10.72)$$

Reflected Wave:

$(\mathbf{E}_r, \mathbf{H}_r)$ is traveling along $-\mathbf{a}_z$ in medium 1. If

$$\mathbf{E}_{rs}(z) = E_{ro}e^{\gamma_1 z} \mathbf{a}_x \quad (10.73)$$

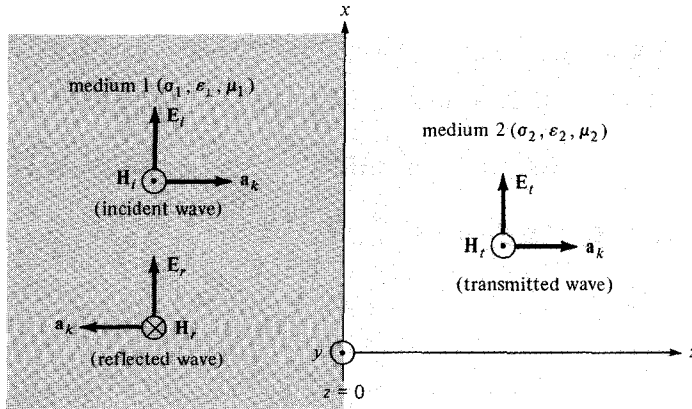


Figure 10.11 A plane wave incident normally on an interface between two different media.

then

$$\mathbf{H}_{rs}(z) = H_{ro} e^{\gamma_1 z} (-\mathbf{a}_y) = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \mathbf{a}_y \quad (10.74)$$

where $\mathbf{E}_{ro}$ has been assumed to be along $\mathbf{a}_x$; we will consistently assume that for normal incident $\mathbf{E}_i$, $\mathbf{E}_r$, and $\mathbf{E}_t$ have the same polarization.

Transmitted Wave:

$(\mathbf{E}_t, \mathbf{H}_t)$ is traveling along $+\mathbf{a}_z$ in medium 2. If

$$\mathbf{E}_{ts}(z) = E_{to} e^{-\gamma_2 z} \mathbf{a}_x \quad (10.75)$$

then

$$\mathbf{H}_{ts}(z) = H_{to} e^{-\gamma_2 z} \mathbf{a}_y = \frac{E_{to}}{\eta_2} e^{-\gamma_2 z} \mathbf{a}_y \quad (10.76)$$

In eqs. (10.71) to (10.76), E_{io} , E_{ro} , and E_{to} are, respectively, the magnitudes of the incident, reflected, and transmitted electric fields at $z = 0$.

Notice from Figure 10.11 that the total field in medium 1 comprises both the incident and reflected fields, whereas medium 2 has only the transmitted field, that is,

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r, \quad \mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$$

$$\mathbf{E}_2 = \mathbf{E}_t, \quad \mathbf{H}_2 = \mathbf{H}_t$$

At the interface $z = 0$, the boundary conditions require that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ fields must be continuous. Since the waves are transverse, $\mathbf{E}$ and $\mathbf{H}$ fields

are entirely tangential to the interface. Hence at $z = 0$, $\mathbf{E}_{1\text{tan}} = \mathbf{E}_{2\text{tan}}$ and $\mathbf{H}_{1\text{tan}} = \mathbf{H}_{2\text{tan}}$ imply that

$$\mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) \quad \rightarrow \quad E_{io} + E_{ro} = E_{to} \quad (10.77)$$

$$\mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) \quad \rightarrow \quad \frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{E_{to}}{\eta_2} \quad (10.78)$$

From eqs. (10.77) and (10.78), we obtain

$$E_{ro} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{io} \quad (10.79)$$

and

$$E_{to} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{io} \quad (10.80)$$

We now define the *reflection coefficient* Γ and the *transmission coefficient* τ from eqs. (10.79) and (10.80) as

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad (10.81a)$$

or

$$E_{ro} = \Gamma E_{io} \quad (10.81b)$$

and

$$\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad (10.82a)$$

or

$$E_{to} = \tau E_{io} \quad (10.82b)$$

Note that

1. $1 + \Gamma = \tau$
2. Both Γ and τ are dimensionless and may be complex.
3. $0 \leq |\Gamma| \leq 1$ (10.83)

The case considered above is the general case. Let us now consider a special case when medium 1 is a perfect dielectric (lossless, $\sigma_1 = 0$) and medium 2 is a perfect conductor ($\sigma_2 = \infty$). For this case, $\eta_2 = 0$; hence, $\Gamma = -1$, and $\tau = 0$, showing that the wave is totally reflected. This should be expected because fields in a perfect conductor must vanish, so there can be no transmitted wave ($\mathbf{E}_2 = 0$). The totally reflected wave combines with the incident wave to form a *standing wave*. A standing wave “stands” and does not

travel; it consists of two traveling waves ($\mathbf{E}_i$ and $\mathbf{E}_r$) of equal amplitudes but in opposite directions. Combining eqs. (10.71) and (10.73) gives the standing wave in medium 1 as

$$\mathbf{E}_{1s} = \mathbf{E}_{is} + \mathbf{E}_{rs} = (E_{io}e^{-\gamma_1 z} + E_{ro}e^{\gamma_1 z}) \mathbf{a}_x \quad (10.84)$$

But

$$\Gamma = \frac{E_{ro}}{E_{io}} = -1, \sigma_1 = 0, \alpha_1 = 0, \gamma_1 = j\beta_1$$

Hence,

$$\mathbf{E}_{1s} = -E_{io}(e^{j\beta_1 z} - e^{-j\beta_1 z}) \mathbf{a}_x$$

or

$$\mathbf{E}_{1s} = -2jE_{io} \sin \beta_1 z \mathbf{a}_x \quad (10.85)$$

Thus

$$\mathbf{E}_1 = \text{Re} (\mathbf{E}_{1s} e^{j\omega t})$$

or

$$\mathbf{E}_1 = 2E_{io} \sin \beta_1 z \sin \omega t \mathbf{a}_x \quad (10.86)$$

By taking similar steps, it can be shown that the magnetic field component of the wave is

$$\mathbf{H}_1 = \frac{2E_{io}}{\eta_1} \cos \beta_1 z \cos \omega t \mathbf{a}_y \quad (10.87)$$

A sketch of the standing wave in eq. (10.86) is presented in Figure 10.12 for $t = 0, T/8, T/4, 3T/8, T/2$, and so on, where $T = 2\pi/\omega$. From the figure, we notice that the wave does not travel but oscillates.

When media 1 and 2 are both lossless we have another special case ($\sigma_1 = 0 = \sigma_2$). In this case, η_1 and η_2 are real and so are Γ and τ . Let us consider the following cases:

CASE A.

If $\eta_2 > \eta_1$, $\Gamma > 0$. Again there is a standing wave in medium 1 but there is also a transmitted wave in medium 2. However, the incident and reflected waves have amplitudes that are not equal in magnitude. It can be shown that the maximum values of $|\mathbf{E}_1|$ occur at

$$-\beta_1 z_{\max} = n\pi$$

or

$$z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}, \quad n = 0, 1, 2, \dots \quad (10.88)$$

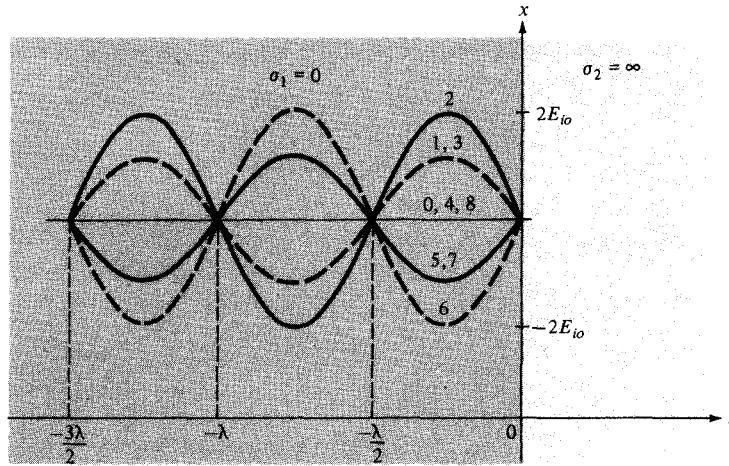


Figure 10.12 Standing waves $E = 2E_{io} \sin \beta_1 z \sin \omega t \mathbf{a}_x$; curves 0, 1, 2, 3, 4, . . . are, respectively, at times $t = 0, T/8, T/4, 3T/8, T/2, \dots$; $\lambda = 2\pi/\beta_1$.

and the minimum values of $|\mathbf{E}_1|$ occur at

$$-\beta_1 z_{\min} = (2n + 1) \frac{\pi}{2}$$

or

$$z_{\min} = -\frac{(2n + 1)\pi}{2\beta_1} = -\frac{(2n + 1)}{4} \lambda_1, \quad n = 0, 1, 2, \dots \quad (10.89)$$

CASE B.

If $\eta_2 < \eta_1$, $\Gamma < 0$. For this case, the locations of $|\mathbf{E}_1|$ maximum are given by eq. (10.89) whereas those of $|\mathbf{E}_1|$ minimum are given by eq. (10.88). All these are illustrated in Figure 10.13. Note that

1. $|\mathbf{H}_1|$ minimum occurs whenever there is $|\mathbf{E}_1|$ maximum and vice versa.
2. The transmitted wave (not shown in Figure 10.13) in medium 2 is a purely traveling wave and consequently there are no maxima or minima in this region.

The ratio of $|\mathbf{E}_1|_{\max}$ to $|\mathbf{E}_1|_{\min}$ (or $|\mathbf{H}_1|_{\max}$ to $|\mathbf{H}_1|_{\min}$) is called the *standing-wave ratio* s ; that is,

$$s = \frac{|\mathbf{E}_1|_{\max}}{|\mathbf{E}_1|_{\min}} = \frac{|\mathbf{H}_1|_{\max}}{|\mathbf{H}_1|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad (10.90)$$

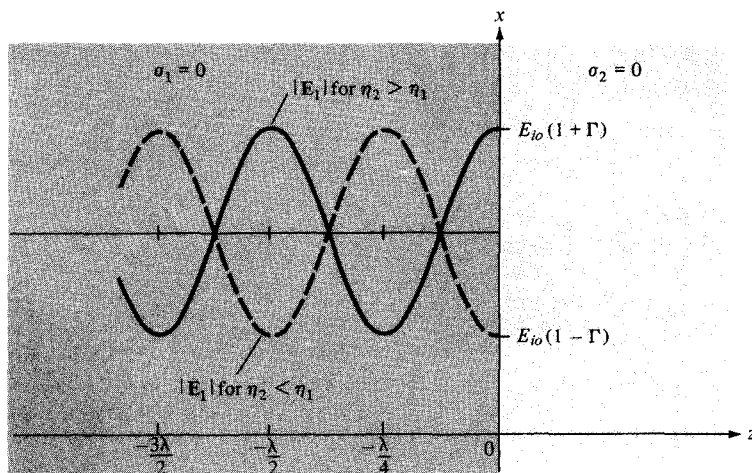


Figure 10.13 Standing waves due to reflection at an interface between two lossless media; $\lambda = 2\pi/\beta_1$.

or

$$|\Gamma| = \frac{s-1}{s+1} \quad (10.91)$$

Since $|\Gamma| \leq 1$, it follows that $1 \leq s \leq \infty$. The standing-wave ratio is dimensionless and it is customarily expressed in decibels (dB) as

$$s \text{ in dB} = 20 \log_{10} s \quad (10.92)$$

EXAMPLE 10.8

In free space ($z \leq 0$), a plane wave with

$$\mathbf{H} = 10 \cos(10^8 t - \beta z) \mathbf{a}_x \text{ mA/m}$$

is incident normally on a lossless medium ($\epsilon = 2\epsilon_0$, $\mu = 8\mu_0$) in region $z \geq 0$. Determine the reflected wave $\mathbf{H}_r$, $\mathbf{E}_r$ and the transmitted wave $\mathbf{H}_t$, $\mathbf{E}_t$.

Solution:

This problem can be solved in two different ways.

Method 1: Consider the problem as illustrated in Figure 10.14. For free space,

$$\beta_1 = \frac{\omega}{c} = \frac{10^8}{3 \times 10^8} = \frac{1}{3}$$

$$\eta_1 = \eta_0 = 120\pi$$

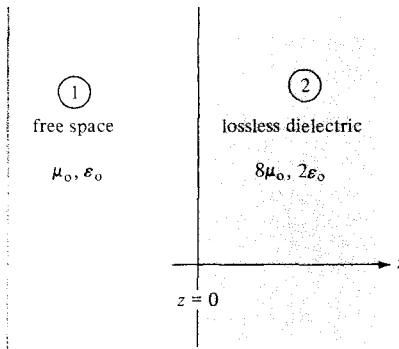


Figure 10.14 For Example 10.8.

For the lossless dielectric medium,

$$\beta_2 = \omega \sqrt{\mu \epsilon} = \omega \sqrt{\mu_0 \epsilon_0} \sqrt{\mu_r \epsilon_r} = \frac{\omega}{c} \cdot (4) = 4\beta_1 = \frac{4}{3}$$

$$\eta_2 = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{\frac{\mu_r}{\epsilon_r}} = 2 \eta_0$$

Given that $\mathbf{H}_i = 10 \cos(10^8 t - \beta_1 z) \mathbf{a}_x$, we expect that

$$\mathbf{E}_i = E_{io} \cos(10^8 t - \beta_1 z) \mathbf{a}_{E_i}$$

where

$$\mathbf{a}_{E_i} = \mathbf{a}_{H_i} \times \mathbf{a}_{k_i} = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

and

$$E_{io} = \eta_1 H_{io} = 10 \eta_0$$

Hence,

$$\mathbf{E}_i = -10\eta_0 \cos(10^8 t - \beta_1 z) \mathbf{a}_y \text{ mV/m}$$

Now

$$\frac{E_{ro}}{E_{io}} = \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_0 - \eta_0}{2\eta_0 + \eta_0} = \frac{1}{3}$$

$$E_{ro} = \frac{1}{3} E_{io}$$

Thus

$$\mathbf{E}_r = -\frac{10}{3} \eta_0 \cos\left(10^8 t + \frac{1}{3} z\right) \mathbf{a}_y \text{ mV/m}$$

from which we easily obtain $\mathbf{H}_r$ as

$$\mathbf{H}_r = -\frac{10}{3} \cos \left(10^8 t + \frac{1}{3} z \right) \mathbf{a}_x \text{ mA/m}$$

Similarly,

$$\frac{E_{to}}{E_{io}} = \tau = 1 + \Gamma = \frac{4}{3} \quad \text{or} \quad E_{to} = \frac{4}{3} E_{io}$$

Thus

$$\mathbf{E}_t = E_{to} \cos (10^8 t - \beta_2 z) \mathbf{a}_{E_t}$$

where $\mathbf{a}_{E_t} = \mathbf{a}_{E_i} = -\mathbf{a}_y$. Hence,

$$\mathbf{E}_t = -\frac{40}{3} \eta_o \cos \left(10^8 t - \frac{4}{3} z \right) \mathbf{a}_y \text{ mV/m}$$

from which we obtain

$$\mathbf{H}_t = \frac{20}{3} \cos \left(10^8 t - \frac{4}{3} z \right) \mathbf{a}_x \text{ mA/m}$$

Method 2: Alternatively, we can obtain $\mathbf{H}_r$ and $\mathbf{H}_t$ directly from $\mathbf{H}_i$ using

$$\frac{H_{ro}}{H_{io}} = -\Gamma \quad \text{and} \quad \frac{H_{to}}{H_{io}} = \tau \frac{\eta_1}{\eta_2}$$

Thus

$$H_{ro} = -\frac{1}{3} H_{io} = -\frac{10}{3}$$

$$H_{to} = \frac{4}{3} \frac{\eta_o}{2\eta_o} \cdot H_{io} = \frac{2}{3} H_{io} = \frac{20}{3}$$

and

$$\mathbf{H}_r = -\frac{10}{3} \cos (10^8 t + \beta_1 z) \mathbf{a}_x \text{ mA/m}$$

$$\mathbf{H}_t = \frac{20}{3} \cos (10^8 t - \beta_2 z) \mathbf{a}_x \text{ mA/m}$$

as previously obtained.

Notice that the boundary conditions at $z = 0$, namely,

$$\mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) = -\frac{40}{3} \eta_o \cos (10^8 t) \mathbf{a}_y$$

and

$$\mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) = \frac{20}{3} \cos(10^8 t) \mathbf{a}_x$$

are satisfied. These boundary conditions can always be used to cross-check $\mathbf{E}$ and $\mathbf{H}$.

PRACTICE EXERCISE 10.8

A 5-GHz uniform plane wave $\mathbf{E}_{is} = 10 e^{-j\beta z} \mathbf{a}_x$ V/m in free space is incident normally on a large plane, lossless dielectric slab ($z > 0$) having $\epsilon = 4\epsilon_0$, $\mu = \mu_0$. Find the reflected wave $\mathbf{E}_{rs}$ and the transmitted wave $\mathbf{E}_{ts}$.

Answer: $-3.333 \exp(j\beta_1 z) \mathbf{a}_x$ V/m, $6.667 \exp(-j\beta_2 z) \mathbf{a}_x$ V/m where $\beta_2 = 2\beta_1 = 200\pi/3$.

EXAMPLE 10.9

Given a uniform plane wave in air as

$$\mathbf{E}_i = 40 \cos(\omega t - \beta z) \mathbf{a}_x + 30 \sin(\omega t - \beta z) \mathbf{a}_y \text{ V/m}$$

- Find $\mathbf{H}_i$.
- If the wave encounters a perfectly conducting plate normal to the z axis at $z = 0$, find the reflected wave $\mathbf{E}_r$ and $\mathbf{H}_r$.
- What are the total $\mathbf{E}$ and $\mathbf{H}$ fields for $z \leq 0$?
- Calculate the time-average Poynting vectors for $z \leq 0$ and $z \geq 0$.

Solution:

- This is similar to the problem in Example 10.3. We may treat the wave as consisting of two waves $\mathbf{E}_{i1}$ and $\mathbf{E}_{i2}$, where

$$\mathbf{E}_{i1} = 40 \cos(\omega t - \beta z) \mathbf{a}_x, \quad \mathbf{E}_{i2} = 30 \sin(\omega t - \beta z) \mathbf{a}_y$$

At atmospheric pressure, air has $\epsilon_r = 1.0006 \approx 1$. Thus air may be regarded as free space. Let $\mathbf{H}_i = \mathbf{H}_{i1} + \mathbf{H}_{i2}$.

$$\mathbf{H}_{i1} = H_{i1o} \cos(\omega t - \beta z) \mathbf{a}_{H_1}$$

where

$$H_{i1o} = \frac{E_{i1o}}{\eta_0} = \frac{40}{120\pi} = \frac{1}{3\pi}$$

$$\mathbf{a}_{H_1} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

Hence

$$\mathbf{H}_{i1} = \frac{1}{3\pi} \cos(\omega t - \beta z) \mathbf{a}_y$$

Similarly,

$$\mathbf{H}_{i2} = H_{i2o} \sin(\omega t - \beta z) \mathbf{a}_{H_2}$$

where

$$H_{i2o} = \frac{E_{i2o}}{\eta_o} = \frac{30}{120\pi} = \frac{1}{4\pi}$$

$$\mathbf{a}_{H_2} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

Hence

$$\mathbf{H}_{i2} = -\frac{1}{4\pi} \sin(\omega t - \beta z) \mathbf{a}_x$$

and

$$\begin{aligned} \mathbf{H}_i &= \mathbf{H}_{i1} + \mathbf{H}_{i2} \\ &= -\frac{1}{4\pi} \sin(\omega t - \beta z) \mathbf{a}_x + \frac{1}{3\pi} \cos(\omega t - \beta z) \mathbf{a}_y \text{ mA/m} \end{aligned}$$

This problem can also be solved using Method 2 of Example 10.3.

(b) Since medium 2 is perfectly conducting,

$$\frac{\sigma_2}{\omega \epsilon_2} \gg 1 \rightarrow \eta_2 \ll \eta_1$$

that is,

$$\Gamma \simeq -1, \quad \tau = 0$$

showing that the incident $\mathbf{E}$ and $\mathbf{H}$ fields are totally reflected.

$$E_{ro} = \Gamma E_{io} = -E_{io}$$

Hence,

$$\mathbf{E}_r = -40 \cos(\omega t + \beta z) \mathbf{a}_x - 30 \sin(\omega t + \beta z) \mathbf{a}_y \text{ V/m}$$

$\mathbf{H}_r$ can be found from $\mathbf{E}_r$ just as we did in part (a) of this example or by using Method 2 of the last example starting with $\mathbf{H}_i$. Whichever approach is taken, we obtain

$$\mathbf{H}_r = \frac{1}{3\pi} \cos(\omega t + \beta z) \mathbf{a}_y - \frac{1}{4\pi} \sin(\omega t + \beta z) \mathbf{a}_x \text{ A/m}$$

(c) The total fields in air

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r \quad \text{and} \quad \mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$$

can be shown to be standing wave. The total fields in the conductor are

$$\mathbf{E}_2 = \mathbf{E}_t = 0, \quad \mathbf{H}_2 = \mathbf{H}_t = 0.$$

(d) For $z \leq 0$,

$$\begin{aligned} \mathcal{P}_{1\text{ave}} &= \frac{|\mathbf{E}_{1s}|^2}{2\eta_1} \mathbf{a}_k = \frac{1}{2\eta_0} [E_{io}^2 \mathbf{a}_z - E_{ro}^2 \mathbf{a}_z] \\ &= \frac{1}{240\pi} [(40^2 + 30^2) \mathbf{a}_z - (40^2 + 30^2) \mathbf{a}_z] \\ &= 0 \end{aligned}$$

For $z \geq 0$,

$$\mathcal{P}_{2\text{ave}} = \frac{|\mathbf{E}_{2s}|^2}{2\eta_2} \mathbf{a}_k = \frac{E_{to}^2}{2\eta_2} \mathbf{a}_z = 0$$

because the whole incident power is reflected.

PRACTICE EXERCISE 10.9

The plane wave $\mathbf{E} = 50 \sin(\omega t - 5x) \mathbf{a}_y$ V/m in a lossless medium ($\mu = 4\mu_0$, $\varepsilon = \varepsilon_0$) encounters a lossy medium ($\mu = \mu_0$, $\varepsilon = 4\varepsilon_0$, $\sigma = 0.1$ mhos/m) normal to the x -axis at $x = 0$. Find

- Γ , τ , and s
- $\mathbf{E}_r$ and $\mathbf{H}_r$
- $\mathbf{E}_t$ and $\mathbf{H}_t$
- The time-average Poynting vectors in both regions

Answer: (a) $0.8186 \angle 171.1^\circ$, $0.2295 \angle 33.56^\circ$, 10.025 , (b) $40.93 \sin(\omega t + 5x + 171.9^\circ) \mathbf{a}_y$ V/m, $-54.3 \sin(\omega t + 5x + 171.9^\circ) \mathbf{a}_z$ mA/m, (c) $11.47 e^{-6.021x} \sin(\omega t - 7.826x + 33.56^\circ) \mathbf{a}_y$ V/m, $120.2 e^{-6.021x} \sin(\omega t - 7.826x - 4.01^\circ) \mathbf{a}_z$ mA/m, (d) $0.5469 \mathbf{a}_x$ W/m², $0.5469 \exp(-12.04x) \mathbf{a}_x$ W/m².

10.9 REFLECTION OF A PLANE WAVE AT OBLIQUE INCIDENCE

We now consider a more general situation than that in Section 10.8. To simplify the analysis, we will assume that we are dealing with lossless media. (We may extend our analysis to that of lossy media by merely replacing ε by ε_c .) It can be shown (see Problems 10.14 and 10.15) that a uniform plane wave takes the general form of

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \\ &= \text{Re} [E_0 e^{j(\mathbf{k} \cdot \mathbf{r} - \omega t)}]\end{aligned}\quad (10.93)$$

where $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the radius or position vector and $\mathbf{k} = k_x\mathbf{a}_x + k_y\mathbf{a}_y + k_z\mathbf{a}_z$ is the *wave number vector* or the *propagation vector*; $\mathbf{k}$ is always in the direction of wave propagation. The magnitude of $\mathbf{k}$ is related to ω according to the dispersion relation

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon \quad (10.94)$$

Thus, for lossless media, k is essentially the same as β in the previous sections. With the general form of $\mathbf{E}$ as in eq. (10.93), Maxwell's equations reduce to

$$\mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H} \quad (10.95a)$$

$$\mathbf{k} \times \mathbf{H} = -\omega \varepsilon \mathbf{E} \quad (10.95b)$$

$$\mathbf{k} \cdot \mathbf{H} = 0 \quad (10.95c)$$

$$\mathbf{k} \cdot \mathbf{E} = 0 \quad (10.95d)$$

showing that (i) $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ are mutually orthogonal, and (ii) $\mathbf{E}$ and $\mathbf{H}$ lie on the plane

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z = \text{constant}$$

From eq. (10.95a), the $\mathbf{H}$ field corresponding to the $\mathbf{E}$ field in eq. (10.93) is

$$\mathbf{H} = \frac{1}{\omega \mu} \mathbf{k} \times \mathbf{E} = \frac{\mathbf{a}_k \times \mathbf{E}}{\eta} \quad (10.96)$$

Having expressed $\mathbf{E}$ and $\mathbf{H}$ in the general form, we can now consider the oblique incidence of a uniform plane wave at a plane boundary as illustrated in Figure 10.15(a). The plane defined by the propagation vector $\mathbf{k}$ and a unit normal vector $\mathbf{a}_n$ to the boundary is called the *plane of incidence*. The angle θ_i between $\mathbf{k}$ and $\mathbf{a}_n$ is the *angle of incidence*.

Again, both the incident and the reflected waves are in medium 1 while the transmitted (or refracted wave) is in medium 2. Let

$$\mathbf{E}_i = \mathbf{E}_{io} \cos(k_{ix}x + k_{iy}y + k_{iz}z - \omega_i t) \quad (10.97a)$$

$$\mathbf{E}_r = \mathbf{E}_{ro} \cos(k_{rx}x + k_{ry}y + k_{rz}z - \omega_r t) \quad (10.97b)$$

$$\mathbf{E}_t = \mathbf{E}_{to} \cos(k_{tx}x + k_{ty}y + k_{tz}z - \omega_t t) \quad (10.97c)$$

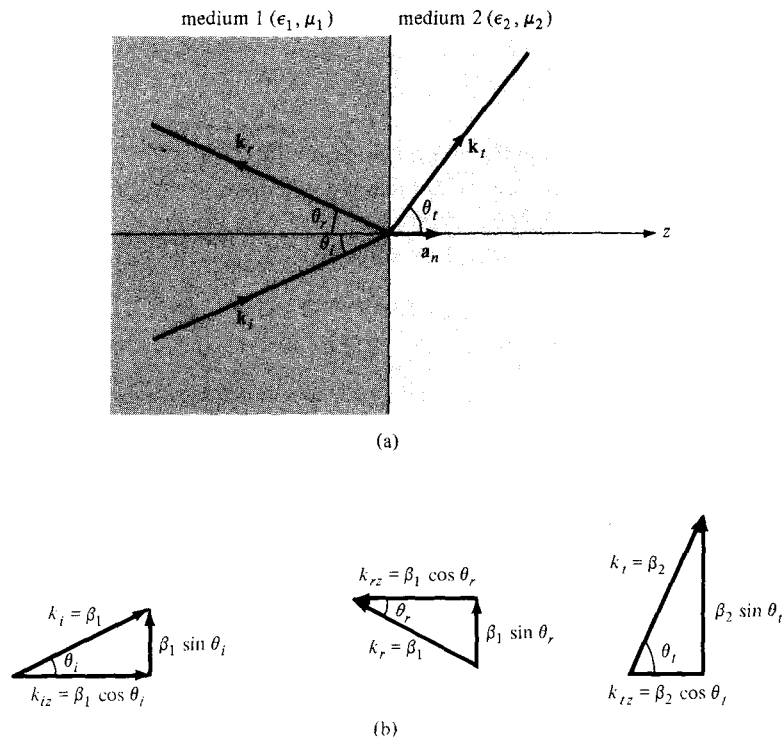


Figure 10.15 Oblique incidence of a plane wave: (a) illustration of θ_i , θ_r , and θ_t ; (b) illustration of the normal and tangential components of $\mathbf{k}$.

where k_i , k_r , and k_t with their normal and tangential components are shown in Figure 10.15(b). Since the tangential component of $\mathbf{E}$ must be continuous at the boundary $z = 0$,

$$\mathbf{E}_i(z = 0) + \mathbf{E}_r(z = 0) = \mathbf{E}_t(z = 0) \quad (10.98)$$

The only way this boundary condition will be satisfied by the waves in eq. (10.97) for all x and y is that

1. $\omega_i = \omega_r = \omega_t = \omega$
2. $k_{ix} = k_{rx} = k_{tx} = k_x$
3. $k_{iy} = k_{ry} = k_{ty} = k_y$

Condition 1 implies that the frequency is unchanged. Conditions 2 and 3 require that the tangential components of the propagation vectors be continuous (called the *phase matching conditions*). This means that the propagation vectors $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ must all lie in the plane of incidence. Thus, by conditions 2 and 3,

$$k_i \sin \theta_i = k_r \sin \theta_r \quad (10.99)$$

$$k_i \sin \theta_i = k_t \sin \theta_t \quad (10.100)$$

where θ_r is the angle of reflection and θ_t is the angle of transmission. But for lossless media,

$$k_i = k_r = \beta_1 = \omega \sqrt{\mu_1 \epsilon_1} \quad (10.101a)$$

$$k_t = \beta_2 = \omega \sqrt{\mu_2 \epsilon_2} \quad (10.101b)$$

From eqs. (10.99) and (10.101a), it is clear that

$$\theta_r = \theta_i \quad (10.102)$$

so that the angle of reflection θ_r equals the angle of incidence θ_i as in optics. Also from eqs. (10.100) and (10.101),

$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{k_i}{k_t} = \frac{\mu_2}{\mu_1} = \sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2}} \quad (10.103)$$

where $u = \omega/k$ is the phase velocity. Equation (10.103) is the well-known *Snell's law*, which can be written as

$$n_1 \sin \theta_i = n_2 \sin \theta_t \quad (10.104)$$

where $n_1 = c\sqrt{\mu_1 \epsilon_1} = c/u_1$ and $n_2 = c\sqrt{\mu_2 \epsilon_2} = c/u_2$ are the *refractive indices* of the media.

Based on these general preliminaries on oblique incidence, we will now specifically consider two special cases: one with the $\mathbf{E}$ field perpendicular to the plane of incidence, the other with the $\mathbf{E}$ field parallel to it. Any other polarization may be considered as a linear combination of these two cases.

A. Parallel Polarization

This case is illustrated in Figure 10.16 where the $\mathbf{E}$ field lies in the xz -plane, the plane of incidence. In medium 1, we have both incident and reflected fields given by

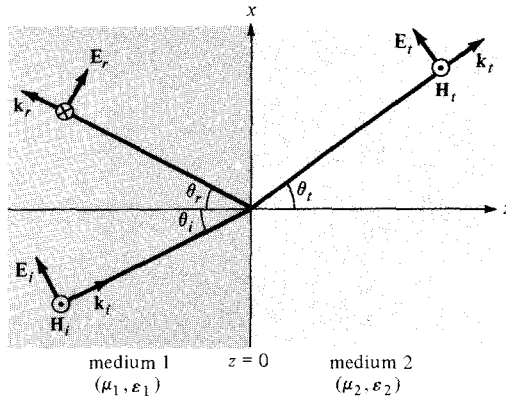
$$\mathbf{E}_{is} = E_{io}(\cos \theta_i \mathbf{a}_x - \sin \theta_i \mathbf{a}_z) e^{-j\beta_1(x \sin \theta_i + z \cos \theta_i)} \quad (10.105a)$$

$$\mathbf{H}_{is} = \frac{E_{io}}{\eta_1} e^{-j\beta_1(x \sin \theta_i + z \cos \theta_i)} \mathbf{a}_y \quad (10.105b)$$

$$\mathbf{E}_{rs} = E_{ro}(\cos \theta_r \mathbf{a}_x + \sin \theta_r \mathbf{a}_z) e^{-j\beta_1(x \sin \theta_r - z \cos \theta_r)} \quad (10.106a)$$

$$\mathbf{H}_{rs} = -\frac{E_{ro}}{\eta_1} e^{-j\beta_1(x \sin \theta_r - z \cos \theta_r)} \mathbf{a}_y \quad (10.106b)$$

where $\beta_1 = \omega \sqrt{\mu_1 \epsilon_1}$. Notice carefully how we arrive at each field component. The trick in deriving the components is to first get the polarization vector $\mathbf{k}$ as shown in Figure 10.15(b) for incident, reflected, and transmitted waves. Once $\mathbf{k}$ is known, we

Figure 10.16 Oblique incidence with $\mathbf{E}$ parallel to the plane of incidence.

define $\mathbf{E}_s$ such that $\nabla \cdot \mathbf{E}_s = 0$ or $\mathbf{k} \cdot \mathbf{E}_s = 0$ and then $\mathbf{H}_s$ is obtained from $\mathbf{H}_s = \frac{\mathbf{k}}{\omega\mu} \times \mathbf{E}_s = \mathbf{a}_k \times \frac{\mathbf{E}}{\eta}$.

The transmitted fields exist in medium 2 and are given by

$$\mathbf{E}_{ts} = E_{to}(\cos \theta_t \mathbf{a}_x - \sin \theta_t \mathbf{a}_z) e^{-j\beta_2(x \sin \theta_t + z \cos \theta_t)} \quad (10.107a)$$

$$\mathbf{H}_{ts} = \frac{E_{to}}{\eta_2} e^{-j\beta_2(x \sin \theta_t + z \cos \theta_t)} \mathbf{a}_y \quad (10.107b)$$

where $\beta_2 = \omega \sqrt{\mu_2 \epsilon_2}$. Should our assumption about the relative directions in eqs. (10.105) to (10.107) be wrong, the final result will show us by means of its sign.

Requiring that $\theta_r = \theta_i$ and that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ be continuous at the boundary $z = 0$, we obtain

$$(E_{io} + E_{ro}) \cos \theta_i = E_{to} \cos \theta_t \quad (10.108a)$$

$$\frac{1}{\eta_1} (E_{io} - E_{ro}) = \frac{1}{\eta_2} E_{to} \quad (10.108b)$$

Expressing E_{ro} and E_{to} in terms of E_{io} , we obtain

$$\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} \quad (10.109a)$$

or

$$E_{ro} = \Gamma_{\parallel} E_{io} \quad (10.109b)$$

and

$$\tau_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos \theta_t}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} \quad (10.110a)$$

or

$$E_{to} = \tau_{\parallel} E_{io} \quad (10.110b)$$

Equations (10.109) and (10.110) are called *Fresnel's equations*. Note that the equations reduce to eqs. (10.81) and (10.82) when $\theta_i = \theta_t = 0$ as expected. Since θ_i and θ_t are related according to Snell's law of eq. (10.103), eqs. (10.109) and (10.110) can be written in terms of θ_i by substituting

$$\cos \theta_t = \sqrt{1 - \sin^2 \theta_t} = \sqrt{1 - (u_2/u_1)^2 \sin^2 \theta_i} \quad (10.111)$$

From eqs. (10.109) and (10.110), it is easily shown that

$$1 + \Gamma_{\parallel} = \tau_{\parallel} \left(\frac{\cos \theta_t}{\cos \theta_i} \right) \quad (10.112)$$

From eq. (10.109a), it is evident that it is possible that $\Gamma_{\parallel} = 0$ because the numerator is the difference of two terms. Under this condition, there is no reflection ($E_{ro} = 0$) and the incident angle at which this takes place is called the *Brewster angle* $\theta_{B_{\parallel}}$. The Brewster angle is also known as the *polarizing angle* because an arbitrarily polarized incident wave will be reflected with only the component of $\mathbf{E}$ perpendicular to the plane of incidence. The Brewster effect is utilized in a laser tube where quartz windows are set at the Brewster angle to control polarization of emitted light. The Brewster angle is obtained by setting $\theta_i = \theta_{B_{\parallel}}$ when $\Gamma_{\parallel} = 0$ in eq. (10.109), that is,

$$\eta_2 \cos \theta_t = \eta_1 \cos \theta_{B_{\parallel}}$$

or

$$\eta_2^2 (1 - \sin^2 \theta_t) = \eta_1^2 (1 - \sin^2 \theta_{B_{\parallel}})$$

Introducing eq. (10.103) or (10.104) gives

$$\sin^2 \theta_{B_{\parallel}} = \frac{1 - \mu_2 \epsilon_1 / \mu_1 \epsilon_2}{1 - (\epsilon_1 / \epsilon_2)^2} \quad (10.113)$$

It is of practical value to consider the case when the dielectric media are not only lossless but nonmagnetic as well—that is, $\mu_1 = \mu_2 = \mu_0$. For this situation, eq. (10.113) becomes

$$\sin^2 \theta_{B_{\parallel}} = \frac{1}{1 + \epsilon_1 / \epsilon_2} \rightarrow \sin \theta_{B_{\parallel}} = \sqrt{\frac{\epsilon_2}{\epsilon_1 + \epsilon_2}}$$

or

$$\tan \theta_{B_{\parallel}} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{n_2}{n_1} \quad (10.114)$$

showing that there is a Brewster angle for any combination of ϵ_1 and ϵ_2 .

B. Perpendicular Polarization

In this case, the $\mathbf{E}$ field is perpendicular to the plane of incidence (the xz -plane) as shown in Figure 10.17. This may also be viewed as the case where $\mathbf{H}$ field is parallel to the plane of incidence. The incident and reflected fields in medium 1 are given by

$$\mathbf{E}_{is} = E_{io} e^{-j\beta_1(x \sin \theta_i + z \cos \theta_i)} \mathbf{a}_y \quad (10.115a)$$

$$\mathbf{H}_{is} = \frac{E_{io}}{\eta_1} (-\cos \theta_i \mathbf{a}_x + \sin \theta_i \mathbf{a}_z) e^{-j\beta_1(x \sin \theta_i + z \cos \theta_i)} \quad (10.115b)$$

$$\mathbf{E}_{rs} = E_{ro} e^{-j\beta_1(x \sin \theta_r - z \cos \theta_r)} \mathbf{a}_y \quad (10.116a)$$

$$\mathbf{H}_{rs} = \frac{E_{ro}}{\eta_1} (\cos \theta_r \mathbf{a}_x + \sin \theta_r \mathbf{a}_z) e^{-j\beta_1(x \sin \theta_r - z \cos \theta_r)} \quad (10.116b)$$

while the transmitted fields in medium 2 are given by

$$\mathbf{E}_{ts} = E_{to} e^{-j\beta_2(x \sin \theta_t + z \cos \theta_t)} \mathbf{a}_y \quad (10.117a)$$

$$\mathbf{H}_{ts} = \frac{E_{to}}{\eta_2} (-\cos \theta_t \mathbf{a}_x + \sin \theta_t \mathbf{a}_z) e^{-j\beta_2(x \sin \theta_t + z \cos \theta_t)} \quad (10.117b)$$

Notice that in defining the field components in eqs. (10.115) to (10.117), Maxwell's equations (10.95) are always satisfied. Again, requiring that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ be continuous at $z = 0$ and setting θ_r equal to θ_i , we get

$$E_{io} + E_{ro} = E_{to} \quad (10.118a)$$

$$\frac{1}{\eta_1} (E_{io} - E_{ro}) \cos \theta_i = \frac{1}{\eta_2} E_{to} \cos \theta_t \quad (10.118b)$$

Expressing E_{ro} and E_{to} in terms of E_{io} leads to

$$\Gamma_{\perp} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \quad (10.119a)$$

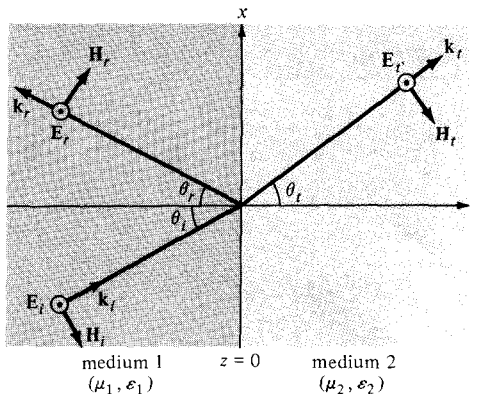


Figure 10.17 Oblique incidence with $\mathbf{E}$ perpendicular to the plane of incidence.

or

$$E_{ro} = \Gamma_{\perp} E_{io} \quad (10.119b)$$

and

$$\tau_{\perp} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \quad (10.120a)$$

or

$$E_{to} = \tau_{\perp} E_{io} \quad (10.120b)$$

which are the *Fresnel's equations* for perpendicular polarization. From eqs. (10.119) and (10.120), it is easy to show that

$$1 + \Gamma_{\perp} = \tau_{\perp} \quad (10.121)$$

which is similar to eq. (10.83) for normal incidence. Also, when $\theta_i = \theta_t = 0$, eqs. (10.119) and (10.120) become eqs. (10.81) and (10.82) as they should.

For no reflection, $\Gamma_{\perp} = 0$ (or $E_r = 0$). This is the same as the case of total transmission ($\tau_{\perp} = 1$). By replacing θ_i with the corresponding Brewster angle $\theta_{B_{\perp}}$, we obtain

$$\eta_2 \cos \theta_{B_{\perp}} = \eta_1 \cos \theta_t$$

or

$$\eta_2^2 (1 - \sin^2 \theta_{B_{\perp}}) = \eta_1^2 (1 - \sin^2 \theta_t)$$

Incorporating eq. (10.104) yields

$$\sin^2 \theta_{B_{\perp}} = \frac{1 - \mu_1 \epsilon_2 / \mu_2 \epsilon_1}{1 - (\mu_1 / \mu_2)^2} \quad (10.122)$$

Note that for nonmagnetic media ($\mu_1 = \mu_2 = \mu_0$), $\sin^2 \theta_{B_{\perp}} \rightarrow \infty$ in eq. (10.122), so $\theta_{B_{\perp}}$ does not exist because the sine of an angle is never greater than unity. Also if $\mu_1 \neq \mu_2$ and $\epsilon_1 = \epsilon_2$, eq. (10.122) reduces to

$$\sin \theta_{B_{\perp}} = \sqrt{\frac{\mu_2}{\mu_1 + \mu_2}}$$

or

$$\tan \theta_{B_{\perp}} = \sqrt{\frac{\mu_2}{\mu_1}} \quad (10.123)$$

Although this situation is theoretically possible, it is rare in practice.

EXAMPLE 10.10

An EM wave travels in free space with the electric field component

$$\mathbf{E}_s = 100 e^{j(0.866y + 0.5z)} \mathbf{a}_x \text{ V/m}$$

Determine

- ω and λ
- The magnetic field component
- The time average power in the wave

Solution:

- Comparing the given $\mathbf{E}$ with

$$\mathbf{E}_s = \mathbf{E}_0 e^{j\mathbf{k} \cdot \mathbf{r}} = E_0 e^{j(k_x x + k_y y + k_z z)} \mathbf{a}_x$$

it is clear that

$$k_x = 0, \quad k_y = 0.866, \quad k_z = 0.5$$

Thus

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{(0.866)^2 + (0.5)^2} = 1$$

But in free space,

$$k = \beta = \omega \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Hence,

$$\omega = kc = 3 \times 10^8 \text{ rad/s}$$

$$\lambda = \frac{2\pi}{k} = 2\pi = 6.283 \text{ m}$$

- From eq. (10.96), the corresponding magnetic field is given by

$$\begin{aligned} \mathbf{H}_s &= \frac{1}{\mu\omega} \mathbf{k} \times \mathbf{E}_s \\ &= \frac{(0.866\mathbf{a}_y + 0.5\mathbf{a}_z)}{4\pi \times 10^{-7} \times 3 \times 10^8} \times 100 \mathbf{a}_x e^{j\mathbf{k} \cdot \mathbf{r}} \end{aligned}$$

or

$$\mathbf{H}_s = (1.33 \mathbf{a}_y - 2.3 \mathbf{a}_z) e^{j(0.866y + 0.5z)} \text{ mA/m}$$

- The time average power is

$$\begin{aligned} \mathcal{P}_{\text{ave}} &= \frac{1}{2} \text{Re} (\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{E_0^2}{2\eta} \mathbf{a}_k \\ &= \frac{(100)^2}{2(120\pi)} (0.866 \mathbf{a}_y + 0.5 \mathbf{a}_z) \\ &= 11.49 \mathbf{a}_y + 6.631 \mathbf{a}_z \text{ W/m}^2 \end{aligned}$$

PRACTICE EXERCISE 10.10

Rework Example 10.10 if

$$\mathbf{E} = (10 \mathbf{a}_y + 5 \mathbf{a}_z) \cos(\omega t + 2y - 4z) \text{ V/m}$$

in free space.

Answer: (a) 1.342×10^9 rad/s, 1.405 m, (b) $-29.66 \cos(1.342 \times 10^9 t + 2y - 4z) \mathbf{a}_x$ mA/m, (c) $-0.07415 \mathbf{a}_y + 0.1489 \mathbf{a}_z$ W/m².

EXAMPLE 10.11

A uniform plane wave in air with

$$\mathbf{E} = 8 \cos(\omega t - 4x - 3z) \mathbf{a}_y \text{ V/m}$$

is incident on a dielectric slab ($z \geq 0$) with $\mu_r = 1.0$, $\epsilon_r = 2.5$, $\sigma = 0$. Find

- (a) The polarization of the wave
- (b) The angle of incidence
- (c) The reflected $\mathbf{E}$ field
- (d) The transmitted $\mathbf{H}$ field

Solution:

(a) From the incident $\mathbf{E}$ field, it is evident that the propagation vector is

$$\mathbf{k}_i = 4\mathbf{a}_x + 3\mathbf{a}_z \rightarrow k_i = 5 = \omega \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c}$$

Hence,

$$\omega = 5c = 15 \times 10^8 \text{ rad/s.}$$

A unit vector normal to the interface ($z = 0$) is $\mathbf{a}_z$. The plane containing $\mathbf{k}$ and $\mathbf{a}_z$ is $y = \text{constant}$, which is the xz -plane, the plane of incidence. Since $\mathbf{E}_i$ is normal to this plane, we have perpendicular polarization (similar to Figure 10.17).

(b) The propagation vectors are illustrated in Figure 10.18 where it is clear that

$$\tan \theta_i = \frac{k_{ix}}{k_{iz}} = \frac{4}{3} \rightarrow \theta_i = 53.13^\circ$$

Alternatively, without Figure 10.18, we can obtain θ_i from the fact that θ_i is the angle between $\mathbf{k}$ and $\mathbf{a}_n$, that is,

$$\cos \theta_i = \mathbf{a}_k \cdot \mathbf{a}_n = \left(\frac{4\mathbf{a}_x + 3\mathbf{a}_z}{5} \right) \cdot \mathbf{a}_z = \frac{3}{5}$$

or

$$\theta_i = 53.13^\circ$$

(c) An easy way to find $\mathbf{E}_r$ is to use eq. (10.116a) because we have noticed that this problem is similar to that considered in Section 10.9(b). Suppose we are not aware of this. Let

$$\mathbf{E}_r = E_{ro} \cos(\omega t - \mathbf{k}_r \cdot \mathbf{r}) \mathbf{a}_y$$

which is similar in form to the given $\mathbf{E}_i$. The unit vector $\mathbf{a}_y$ is chosen in view of the fact that the tangential component of $\mathbf{E}$ must be continuous at the interface. From Figure 10.18,

$$\mathbf{k}_r = k_{rx} \mathbf{a}_x - k_{rz} \mathbf{a}_z$$

where

$$k_{rx} = k_r \sin \theta_r, \quad k_{rz} = k_r \cos \theta_r$$

But $\theta_r = \theta_i$ and $k_r = k_i = 5$ because both k_r and k_i are in the same medium. Hence,

$$\mathbf{k}_r = 4\mathbf{a}_x - 3\mathbf{a}_z$$

To find E_{ro} , we need θ_t . From Snell's law

$$\begin{aligned} \sin \theta_t &= \frac{n_1}{n_2} \sin \theta_i = \frac{c\sqrt{\mu_1\epsilon_1}}{c\sqrt{\mu_2\epsilon_2}} \sin \theta_i \\ &= \frac{\sin 53.13^\circ}{\sqrt{2.5}} \end{aligned}$$

or

$$\theta_t = 30.39^\circ$$

$$\begin{aligned} \Gamma_\perp &= \frac{E_{ro}}{E_{io}} \\ &= \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \end{aligned}$$

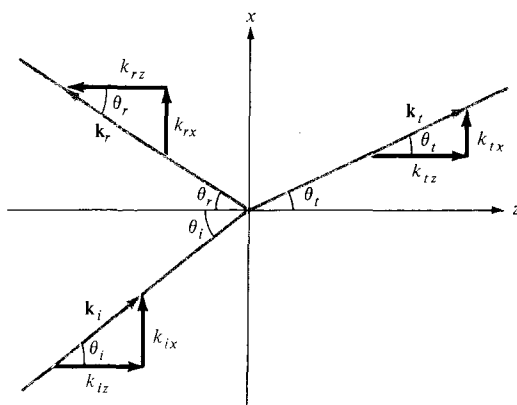


Figure 10.18 Propagation vectors of Example 10.11.

where $\eta_1 = \eta_0 = 377$, $\eta_2 = \sqrt{\frac{\mu_0 \mu_{r_2}}{\epsilon_0 \epsilon_{r_2}}} = \frac{377}{\sqrt{2.5}} = 238.4$

$$\Gamma_{\perp} = \frac{238.4 \cos 35.13^\circ - 377 \cos 30.39^\circ}{238.4 \cos 53.13^\circ + 377 \cos 30.39^\circ} = -0.389$$

Hence,

$$E_{ro} = \Gamma_{\perp} E_{io} = -0.389 (8) = -3.112$$

and

$$\mathbf{E}_r = -3.112 \cos (15 \times 10^8 t - 4x + 3z) \mathbf{a}_y \text{ V/m}$$

(d) Similarly, let the transmitted electric field be

$$\mathbf{E}_t = E_{to} \cos (\omega t - \mathbf{k}_t \cdot \mathbf{r}) \mathbf{a}_y$$

where

$$\begin{aligned} k_t &= \beta_2 = \omega \sqrt{\mu_2 \epsilon_2} = \frac{\omega}{c} \sqrt{\mu_{r_2} \epsilon_{r_2}} \\ &= \frac{15 \times 10^8}{3 \times 10^8} \sqrt{1 \times 2.5} = 7.906 \end{aligned}$$

From Figure 10.18,

$$k_{tx} = k_t \sin \theta_t = 4$$

$$k_{tz} = k_t \cos \theta_t = 6.819$$

or

$$\mathbf{k}_t = 4\mathbf{a}_x + 6.819\mathbf{a}_z$$

Notice that $k_{ix} = k_{rx} = k_{tx}$ as expected.

$$\begin{aligned} \tau_{\perp} &= \frac{E_{to}}{E_{io}} = \frac{2 \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \\ &= \frac{2 \times 238.4 \cos 53.13^\circ}{238.4 \cos 53.13^\circ + 377 \cos 30.39^\circ} \\ &= 0.611 \end{aligned}$$

The same result could be obtained from the relation $\tau_{\perp} = 1 + \Gamma_{\perp}$. Hence,

$$E_{to} = \tau_{\perp} E_{io} = 0.611 \times 8 = 4.888$$

$$\mathbf{E}_t = 4.888 \cos (15 \times 10^8 t - 4x - 6.819z) \mathbf{a}_y$$

From $\mathbf{E}_t$, $\mathbf{H}_t$ is easily obtained as

$$\begin{aligned}\mathbf{H}_t &= \frac{1}{\mu_2 \omega} \mathbf{k}_t \times \mathbf{E}_t = \frac{\mathbf{a}_{k_t} \times \mathbf{E}_t}{\eta_2} \\ &= \frac{4\mathbf{a}_x + 6.819\mathbf{a}_z}{7.906 (238.4)} \times 4.888 \mathbf{a}_y \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \\ \mathbf{H}_t &= (-17.69 \mathbf{a}_x + 10.37 \mathbf{a}_z) \cos(15 \times 10^8 t - 4x - 6.819z) \text{ mA/m}.\end{aligned}$$

PRACTICE EXERCISE 10.11

If the plane wave of Practice Exercise 10.10 is incident on a dielectric medium having $\sigma = 0$, $\epsilon = 4\epsilon_0$, $\mu = \mu_0$ and occupying $z \geq 0$, calculate

- The angles of incidence, reflection, and transmission
- The reflection and transmission coefficients
- The total $\mathbf{E}$ field in free space
- The total $\mathbf{E}$ field in the dielectric
- The Brewster angle.

Answer: (a) 26.56° , 26.56° , 12.92° , (b) -0.295 , 0.647 , (c) $(10\mathbf{a}_y + 5\mathbf{a}_z) \cos(\omega t + 2y - 4z) + (-2.946\mathbf{a}_y + 1.473\mathbf{a}_z) \cos(\omega t + 2y + 4z)$ V/m, (d) $(7.055\mathbf{a}_y + 1.618\mathbf{a}_z) \cos(\omega t + 2y - 8.718z)$ V/m, (e) 63.43° .

SUMMARY 1. The wave equation is of the form

$$\frac{\partial^2 \Phi}{\partial t^2} - u^2 \frac{\partial^2 \Phi}{\partial z^2} = 0$$

with the solution

$$\Phi = A \sin(\omega t - \beta z)$$

where u = wave velocity, A = wave amplitude, ω = angular frequency ($=2\pi f$), and β = phase constant. Also, $\beta = \omega/u = 2\pi/\lambda$ or $u = f\lambda = \lambda/T$, where λ = wavelength and T = period.

- In a lossy, charge-free medium, the wave equation based on Maxwell's equations is of the form

$$\nabla^2 \mathbf{A}_s - \gamma^2 \mathbf{A}_s = 0$$

where $\mathbf{A}_s$ is either $\mathbf{E}_s$ or $\mathbf{H}_s$ and $\gamma = \alpha + j\beta$ is the propagation constant. If we assume $\mathbf{E}_s = E_{xs}(z) \mathbf{a}_x$, we obtain EM waves of the form

$$\mathbf{E}(z, t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \mathbf{a}_x$$

$$\mathbf{H}(z, t) = H_0 e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \mathbf{a}_y$$

where α = attenuation constant, β = phase constant, $\eta = |\eta|/\angle\eta$ = intrinsic impedance of the medium. The reciprocal of α is the skin depth ($\delta = 1/\alpha$). The relationship between β , ω , and λ as stated above remain valid for EM waves.

3. Wave propagation in other types of media can be derived from that for lossy media as special cases. For free space, set $\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$; for lossless dielectric media, set $\sigma = 0$, $\epsilon = \epsilon_0\epsilon_r$, and $\mu = \mu_0\mu_r$; and for good conductors, set $\sigma \approx \infty$, $\epsilon = \epsilon_0$, $\mu = \mu_0$, or $\sigma/\omega\epsilon \rightarrow 0$.
4. A medium is classified as lossy dielectric, lossless dielectric or good conductor depending on its loss tangent given by

$$\tan \theta = \frac{|\mathbf{J}_s|}{|\mathbf{J}_{ds}|} = \frac{\sigma}{\omega\epsilon} = \frac{\epsilon''}{\epsilon'}$$

where $\epsilon_c = \epsilon' - j\epsilon''$ is the complex permittivity of the medium. For lossless dielectrics $\tan \theta \ll 1$, for good conductors $\tan \theta \gg 1$, and for lossy dielectrics $\tan \theta$ is of the order of unity.

5. In a good conductor, the fields tend to concentrate within the initial distance δ from the conductor surface. This phenomenon is called skin effect. For a conductor of width w and length ℓ , the effective or ac resistance is

$$R_{ac} = \frac{\ell}{\sigma w \delta}$$

where δ is the skin depth.

6. The Poynting vector, $\mathcal{P}$, is the power-flow vector whose direction is the same as the direction of wave propagation and magnitude the same as the amount of power flowing through a unit area normal to its direction.

$$\mathcal{P} = \mathbf{E} \times \mathbf{H}, \quad \mathcal{P}_{ave} = 1/2 \operatorname{Re} (\mathbf{E}_s \times \mathbf{H}_s^*)$$

7. If a plane wave is incident normally from medium 1 to medium 2, the reflection coefficient Γ and transmission coefficient τ are given by

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad \tau = \frac{E_{to}}{E_{io}} = 1 + \Gamma$$

The standing wave ratio, s , is defined as

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

8. For oblique incidence from lossless medium 1 to lossless medium 2, we have the Fresnel coefficients as

$$\Gamma_{\parallel} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}, \quad \tau_{\parallel} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

for parallel polarization and

$$\Gamma_{\perp} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \quad \tau_{\perp} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

for perpendicular polarization. As in optics,

$$\theta_r = \theta_i$$

$$\frac{\sin \theta_t}{\sin \theta_i} = \frac{\beta_1}{\beta_2} = \sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2}}$$

Total transmission or no reflection ($\Gamma = 0$) occurs when the angle of incidence θ_i is equal to the Brewster angle.

REVIEW QUESTIONS

10.1 Which of these is not a correct form of the wave $E_x = \cos(\omega t - \beta z)$?

- (a) $\cos(\beta z - \omega t)$
- (b) $\sin(\beta z - \omega t - \pi/2)$
- (c) $\cos\left(\frac{2\pi t}{T} - \frac{2\pi z}{\lambda}\right)$
- (d) $\text{Re}(e^{j(\omega t - \beta z)})$
- (e) $\cos \beta(z - ut)$

10.2 Identify which of these functions do not satisfy the wave equation:

- (a) $50e^{j\omega(t-3z)}$
- (b) $\sin \omega(10z + 5t)$
- (c) $(x + 2t)^2$
- (d) $\cos^2(y + 5t)$
- (e) $\sin x \cos t$
- (f) $\cos(5y + 2x)$

10.3 Which of the following statements is not true of waves in general?

- (a) It may be a function of time only.
- (b) It may be sinusoidal or cosinusoidal.
- (c) It must be a function of time and space.
- (d) For practical reasons, it must be finite in extent.

10.4 The electric field component of a wave in free space is given by $\mathbf{E} = 10 \cos(10^7 t + kz) \mathbf{a}_y$ V/m. It can be inferred that

- (a) The wave propagates along $\mathbf{a}_y$.
- (b) The wavelength $\lambda = 188.5$ m.

- (c) The wave amplitude is 10 V/m.
 - (d) The wave number $k = 0.33$ rad/m.
 - (e) The wave attenuates as it travels.
- 10.5** Given that $\mathbf{H} = 0.5 e^{-0.1x} \sin(10^6 t - 2x) \mathbf{a}_z$ A/m, which of these statements are incorrect?
- (a) $\alpha = 0.1$ Np/m
 - (b) $\beta = -2$ rad/m
 - (c) $\omega = 10^6$ rad/s
 - (d) The wave travels along $\mathbf{a}_x$.
 - (e) The wave is polarized in the z -direction.
 - (f) The period of the wave is $1 \mu\text{s}$.
- 10.6** What is the major factor for determining whether a medium is free space, lossless dielectric, lossy dielectric, or good conductor?
- (a) Attenuation constant
 - (b) Constitutive parameters (σ, ϵ, μ)
 - (c) Loss tangent
 - (d) Reflection coefficient
- 10.7** In a certain medium, $\mathbf{E} = 10 \cos(10^8 t - 3y) \mathbf{a}_x$ V/m. What type of medium is it?
- (a) Free space
 - (b) Perfect dielectric
 - (c) Lossless dielectric
 - (d) Perfect conductor
- 10.8** Electromagnetic waves travel faster in conductors than in dielectrics.
- (a) True
 - (b) False
- 10.9** In a good conductor, $\mathbf{E}$ and $\mathbf{H}$ are in time phase.
- (a) True
 - (b) False
- 10.10** The Poynting vector physically denotes the power density leaving or entering a given volume in a time-varying field.
- (a) True
 - (b) False

Answers: 10.1b, 10.2d,f, 10.3a, 10.4b,c, 10.5b,f, 10.6c, 10.7c, 10.8b, 10.9b, 10.10a.

PROBLEMS

- 10.1**
- An EM wave propagating in a certain medium is described by

$$\mathbf{E} = 25 \sin(2\pi \times 10^6 t - 6x) \mathbf{a}_z \text{ V/m}$$

- (a) Determine the direction of wave propagation.
- (b) Compute the period T , the wavelength λ , and the velocity u .
- (c) Sketch the wave at $t = 0, T/8, T/4, T/2$.

- 10.2** (a) Derive eqs. (10.23) and (10.24) from eqs. (10.18) and (10.20).
 (b) Using eq. (10.29) in conjunction with Maxwell's equations, show that

$$\eta = \frac{j\omega\mu}{\gamma}$$

- (c) From part (b), derive eqs. (10.32) and (10.33).

- 10.3** At 50 MHz, a lossy dielectric material is characterized by $\epsilon = 3.6\epsilon_0$, $\mu = 2.1\mu_0$, and $\sigma = 0.08 \text{ S/m}$. If $\mathbf{E}_s = 6e^{-\gamma x} \mathbf{a}_z \text{ V/m}$, compute: (a) γ , (b) λ , (c) u , (d) η , (e) $\mathbf{H}_s$.

- 10.4** A lossy material has $\mu = 5\mu_0$, $\epsilon = 2\epsilon_0$. If at 5 MHz, the phase constant is 10 rad/m, calculate

- (a) The loss tangent
- (b) The conductivity of the material
- (c) The complex permittivity
- (d) The attenuation constant
- (e) The intrinsic impedance

- *10.5** A nonmagnetic medium has an intrinsic impedance $240 \angle 30^\circ \Omega$. Find its

- (a) Loss tangent
- (b) Dielectric constant
- (c) Complex permittivity
- (d) Attenuation constant at 1 MHz

- 10.6** The amplitude of a wave traveling through a lossy nonmagnetic medium reduces by 18% every meter. If the wave operates at 10 MHz and the electric field leads the magnetic field by 24° , calculate: (a) the propagation constant, (b) the wavelength, (c) the skin depth, (d) the conductivity of the medium.

- 10.7** Sea water plays a vital role in the study of submarine communications. Assuming that for sea water, $\sigma = 4 \text{ S/m}$, $\epsilon_r = 80$, $\mu_r = 1$, and $f = 100 \text{ MHz}$, calculate: (a) the phase velocity, (b) the wavelength, (c) the skin depth, (d) the intrinsic impedance.

- 10.8** In a certain medium with $\mu = \mu_0$, $\epsilon = 4\epsilon_0$,

$$\mathbf{H} = 12e^{-0.1y} \sin(\pi \times 10^8 t - \beta y) \mathbf{a}_x \text{ A/m}$$

find: (a) the wave period T , (b) the wavelength λ , (c) the electric field $\mathbf{E}$, (d) the phase difference between $\mathbf{E}$ and $\mathbf{H}$.

10.9 In a medium,

$$\mathbf{E} = 16e^{-0.05x} \sin(2 \times 10^8 t - 2x) \mathbf{a}_z \text{ V/m}$$

find: (a) the propagation constant, (b) the wavelength, (c) the speed of the wave, (d) the skin depth.

10.10 A uniform wave in air has

$$\mathbf{E} = 10 \cos(2\pi \times 10^6 t - \beta z) \mathbf{a}_y$$

- (a) Calculate β and λ .
- (b) Sketch the wave at $z = 0, \lambda/4$.
- (c) Find $\mathbf{H}$.

10.11 The magnetic field component of an EM wave propagating through a nonmagnetic medium ($\mu = \mu_0$) is

$$\mathbf{H} = 25 \sin(2 \times 10^8 t + 6x) \mathbf{a}_y \text{ mA/m}$$

Determine:

- (a) The direction of wave propagation.
- (b) The permittivity of the medium.
- (c) The electric field intensity.

10.12 If $\mathbf{H} = 10 \sin(\omega t - 4z) \mathbf{a}_x$ mA/m in a material for which $\sigma = 0$, $\mu = \mu_0$, $\epsilon = 4\epsilon_0$, calculate ω , λ , and $\mathbf{J}_d$.

10.13 A manufacturer produces a ferrite material with $\mu = 750\mu_0$, $\epsilon = 5\epsilon_0$, and $\sigma = 10^{-6}$ S/m at 10 MHz.

- (a) Would you classify the material as lossless, lossy, or conducting?
- (b) Calculate β and λ .
- (c) Determine the phase difference between two points separated by 2 m.
- (d) Find the intrinsic impedance.

***10.14** By assuming the time-dependent fields $\mathbf{E} = \mathbf{E}_0 e^{j(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ and $\mathbf{H} = \mathbf{H}_0 e^{j(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ where $\mathbf{k} = k_x \mathbf{a}_x + k_y \mathbf{a}_y + k_z \mathbf{a}_z$ is the wave number vector and $\mathbf{r} = x \mathbf{a}_x + y \mathbf{a}_y + z \mathbf{a}_z$ is the radius vector, show that $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ can be expressed as $\mathbf{k} \times \mathbf{E} = \mu \omega \mathbf{H}$ and deduce $\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$.

10.15 Assume the same fields as in Problem 10.14 and show that Maxwell's equations in a source-free region can be written as

$$\mathbf{k} \cdot \mathbf{E} = 0$$

$$\mathbf{k} \cdot \mathbf{H} = 0$$

$$\mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}$$

$$\mathbf{k} \times \mathbf{H} = -\omega \epsilon \mathbf{E}$$

From these equations deduce

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H \quad \text{and} \quad \mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

10.16 The magnetic field component of a plane wave in a lossless dielectric is

$$\mathbf{H} = 30 \sin(2\pi \times 10^8 t - 5x) \mathbf{a}_z \text{ mA/m}$$

- If $\mu_r = 1$, find ϵ_r .
- Calculate the wavelength and wave velocity.
- Determine the wave impedance.
- Determine the polarization of the wave.
- Find the corresponding electric field component.
- Find the displacement current density.

10.17 In a nonmagnetic medium,

$$\mathbf{E} = 50 \cos(10^9 t - 8x) \mathbf{a}_y + 40 \sin(10^9 t - 8x) \mathbf{a}_z \text{ V/m}$$

find the dielectric constant ϵ_r and the corresponding $\mathbf{H}$.

10.18 In a certain medium

$$\mathbf{E} = 10 \cos(2\pi \times 10^7 t - \beta x)(\mathbf{a}_y + \mathbf{a}_z) \text{ V/m}$$

If $\mu = 50\mu_0$, $\epsilon = 2\epsilon_0$, and $\sigma = 0$, find β and $\mathbf{H}$.

10.19 Which of the following media may be treated as conducting at 8 MHz?

- Wet marshy soil ($\epsilon = 15\epsilon_0$, $\mu = \mu_0$, $\sigma = 10^{-2} \text{ S/m}$)
- Intrinsic germanium ($\epsilon = 16\epsilon_0$, $\mu = \mu_0$, $\sigma = 0.025 \text{ S/m}$)
- Sea water ($\epsilon = 81\epsilon_0$, $\mu = \mu_0$, $\sigma = 25 \text{ S/m}$)

10.20 Calculate the skin depth and the velocity of propagation for a uniform plane wave at frequency 6 MHz traveling in polyvinylchloride ($\mu_r = 1$, $\epsilon_r = 4$, $\tan \theta_\eta = 7 \times 10^{-2}$).

10.21 A uniform plane wave in a lossy medium has a phase constant of 1.6 rad/m at 10^7 Hz and its magnitude is reduced by 60% for every 2 m traveled. Find the skin depth and speed of the wave.

10.22 (a) Determine the dc resistance of a round copper wire ($\sigma = 5.8 \times 10^7 \text{ S/m}$, $\mu_r = 1$, $\epsilon_r = 1$) of radius 1.2 mm and length 600 m.

(b) Find the ac resistance at 100 MHz.

(c) Calculate the approximate frequency where dc and ac resistances are equal.

10.23 A 40-m-long aluminum ($\sigma = 3.5 \times 10^7 \text{ S/m}$, $\mu_r = 1$, $\epsilon_r = 1$) pipe with inner and outer radii 9 mm and 12 mm carries a total current of $6 \sin 10^6 \pi t \text{ A}$. Find the skin depth and the effective resistance of the pipe.

10.24 Show that in a good conductor, the skin depth δ is always much shorter than the wavelength.

10.25 Brass waveguides are often silver plated to reduce losses. If at least the thickness of silver ($\mu = \mu_0$, $\varepsilon = \varepsilon_0$, $\sigma = 6.1 \times 10^7$ S/m) must be 5δ , find the minimum thickness required for a waveguide operating at 12 GHz.

10.26 A uniform plane wave in a lossy nonmagnetic media has

$$\mathbf{E}_s = (5\mathbf{a}_x + 12\mathbf{a}_y)e^{-\gamma z}, \gamma = 0.2 + j3.4/\text{m}$$

- (a) Compute the magnitude of the wave at $z = 4$ m.
- (b) Find the loss in dB suffered by the wave in the interval $0 < z < 3$ m.
- (c) Calculate the Poynting vector at $z = 4$, $t = T/8$. Take $\omega = 10^8$ rad/s.

10.27 In a nonmagnetic material,

$$\mathbf{H} = 30 \cos(2\pi \times 10^8 t - 6x) \mathbf{a}_y \text{ mA/m}$$

find: (a) the intrinsic impedance, (b) the Poynting vector, (c) the time-average power crossing the surface $x = 1$, $0 < y < 2$, $0 < z < 3$ m.

***10.28** Show that eqs. (10.67) and (10.68) are equivalent.

10.29 In a transmission line filled with a lossless dielectric ($\varepsilon = 4.5\varepsilon_0$, $\mu = \mu_0$),

$$\mathbf{E} = \frac{40}{\rho} \sin(\omega t - 2z) \mathbf{a}_\rho \text{ V/m}$$

find: (a) ω and $\mathbf{H}$, (b) the Poynting vector, (c) the total time-average power crossing the surface $z = 1$ m, $2 \text{ mm} < \rho < 3 \text{ mm}$, $0 < \phi < 2\pi$.

10.30 (a) For a normal incidence upon the dielectric–dielectric interface for which $\mu_1 = \mu_2 = \mu_0$, we define R and T as the reflection and transmission coefficients for average powers, i.e., $P_{r,\text{ave}} = RP_{i,\text{ave}}$ and $P_{t,\text{ave}} = TP_{i,\text{ave}}$. Prove that

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \quad \text{and} \quad T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

where n_1 and n_2 are the reflective indices of the media.

(b) Determine the ratio n_1/n_2 so that the reflected and the transmitted waves have the same average power.

10.31 The plane wave $\mathbf{E} = 30 \cos(\omega t - z) \mathbf{a}_x$ V/m in air normally hits a lossless medium ($\mu = \mu_0$, $\varepsilon = 4\varepsilon_0$) at $z = 0$. (a) Find Γ , τ , and s . (b) Calculate the reflected electric and magnetic fields.

10.32 A uniform plane wave in air with

$$\mathbf{H} = 4 \sin(\omega t - 5x) \mathbf{a}_y \text{ A/m}$$

is normally incident on a plastic region with the parameters $\mu = \mu_0$, $\varepsilon = 4\varepsilon_0$, and $\sigma = 0$. (a) Obtain the total electric field in air. (b) Calculate the time-average power density in the plastic region. (c) Find the standing wave ratio.

10.33 A plane wave in free space with $\mathbf{E} = 3.6 \cos(\omega t - 3x) \mathbf{a}_y$ V/m is incident normally on an interface at $x = 0$. If a lossless medium with $\sigma = 0$, $\epsilon_r = 12.5$ exists for $x \geq 0$ and the reflected wave has $\mathbf{H}_r = -1.2 \cos(\omega t + 3x) \mathbf{a}_z$ mA/m, find μ_2 .

10.34 Region 1 is a lossless medium for which $y \geq 0$, $\mu = \mu_0$, $\epsilon = 4\epsilon_0$, whereas region 2 is free space, $y \leq 0$. If a plane wave $\mathbf{E} = 5 \cos(10^8 t + \beta y) \mathbf{a}_z$ V/m exists in region 1, find: (a) the total electric field component of the wave in region 2, (b) the time-average Poynting vector in region 1, (c) the time-average Poynting vector in region 2.

10.35 A plane wave in free space ($z \leq 0$) is incident normally on a large block of material with $\epsilon_r = 12$, $\mu_r = 3$, $\sigma = 0$ which occupies $z \geq 0$. If the incident electric field is

$$\mathbf{E} = 30 \cos(\omega t - z) \mathbf{a}_y \text{ V/m}$$

find: (a) ω , (b) the standing wave ratio, (c) the reflected magnetic field, (d) the average power density of the transmitted wave.

10.36 A 30-MHz uniform plane wave with

$$\mathbf{H} = 10 \sin(\omega t + \beta x) \mathbf{a}_z \text{ mA/m}$$

exists in region $x \geq 0$ having $\sigma = 0$, $\epsilon = 9\epsilon_0$, $\mu = 4\mu_0$. At $x = 0$, the wave encounters free space. Determine (a) the polarization of the wave, (b) the phase constant β , (c) the displacement current density in region $x \geq 0$, (d) the reflected and transmitted magnetic fields, and (e) the average power density in each region.

10.37 A uniform plane wave in air is normally incident on an infinite lossless dielectric material having $\epsilon = 3\epsilon_0$ and $\mu = \mu_0$. If the incident wave is $\mathbf{E}_i = 10 \cos(\omega t - z) \mathbf{a}_y$ V/m, find:

- (a) λ and ω of the wave in air and the transmitted wave in the dielectric medium
- (b) The incident $\mathbf{H}_i$ field
- (c) Γ and τ
- (d) The total electric field and the time-average power in both regions

***10.38** A signal in air ($z \geq 0$) with the electric field component

$$\mathbf{E} = 10 \sin(\omega t + 3z) \mathbf{a}_x \text{ V/m}$$

hits normally the ocean surface at $z = 0$ as in Figure 10.19. Assuming that the ocean surface is smooth and that $\epsilon = 80\epsilon_0$, $\mu = \mu_0$, $\sigma = 4$ mhos/m in ocean, determine

- (a) ω
- (b) The wavelength of the signal in air
- (c) The loss tangent and intrinsic impedance of the ocean
- (d) The reflected and transmitted E field

10.39 Sketch the standing wave in eq. (10.87) at $t = 0, T/8, T/4, 3T/8, T/2$, and so on, where $T = 2\pi/\omega$.

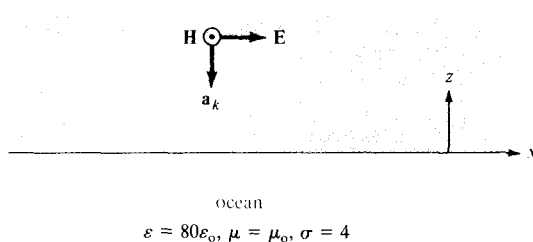


Figure 10.19 For Problem 10.38.

- 10.40** A uniform plane wave is incident at an angle $\theta_i = 45^\circ$ on a pair of dielectric slabs joined together as shown in Figure 10.20. Determine the angles of transmission θ_{t1} and θ_{t2} in the slabs.

- 10.41** Show that the field

$$\mathbf{E}_s = 20 \sin(k_x x) \cos(k_y y) \mathbf{a}_z$$

where $k_x^2 + k_y^2 = \omega^2 \mu_0 \epsilon_0$, can be represented as the superposition of four propagating plane waves. Find the corresponding $\mathbf{H}_s$.

- 10.42** Show that for nonmagnetic dielectric media, the reflection and transmission coefficients for oblique incidence become

$$\Gamma_{\parallel} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}, \quad \tau_{\parallel} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_t + \theta_i) \cos(\theta_t - \theta_i)}$$

$$\Gamma_{\perp} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}, \quad \tau_{\perp} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_t + \theta_i)}$$

- *10.43** A parallel-polarized wave in air with

$$\mathbf{E} = (8\mathbf{a}_y - 6\mathbf{a}_z) \sin(\omega t - 4y - 3z) \text{ V/m}$$

impinges a dielectric half-space as shown in Figure 10.21. Find: (a) the incidence angle θ_i , (b) the time average in air ($\mu = \mu_0$, $\epsilon = \epsilon_0$), (c) the reflected and transmitted $\mathbf{E}$ fields.

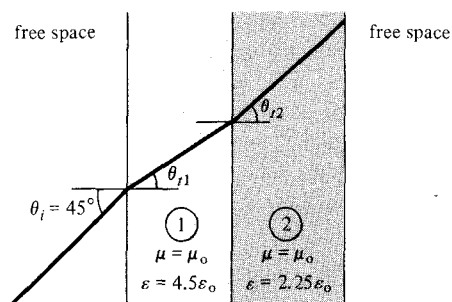


Figure 10.20 For Problem 10.40.

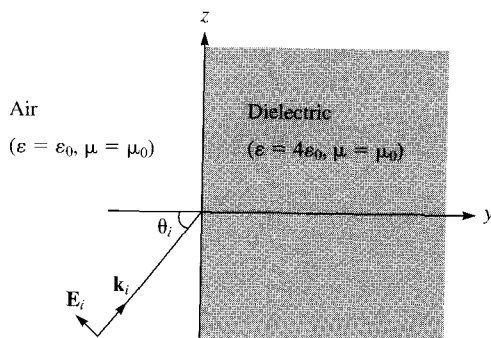


Figure 10.21 For Problem 10.43.

10.44 In a dielectric medium ($\epsilon = 9\epsilon_0$, $\mu = \mu_0$), a plane wave with

$$\mathbf{H} = 0.2 \cos(10^9 t - kx - k\sqrt{8}z) \mathbf{a}_y \text{ A/m}$$

is incident on an air boundary at $z = 0$, find

- θ_r and θ_i
- k
- The wavelength in the dielectric and air
- The incident $\mathbf{E}$
- The transmitted and reflected $\mathbf{E}$
- The Brewster angle

***10.45** A plane wave in air with

$$\mathbf{E} = (8\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z) \sin(\omega t + 3x - 4y) \text{ V/m}$$

is incident on a copper slab in $y \geq 0$. Find ω and the reflected wave. Assume copper is a perfect conductor. (*Hint:* Write down the field components in both media and match the boundary conditions.)

10.46 A polarized wave is incident from air to polystyrene with $\mu = \mu_0$, $\epsilon = 2.6\epsilon_0$ at Brewster angle. Determine the transmission angle.