

PART 4

WAVES AND
APPLICATIONS

Chapter 9

MAXWELL'S EQUATIONS

Do you want to be a hero? Don't be the kind of person who watches others do great things or doesn't know what's happening. Go out and make things happen. The people who get things done have a burning desire to make things happen, get ahead, serve more people, become the best they can possibly be, and help improve the world around them.

—GLENN VAN EKEREN

9.1 INTRODUCTION

In Part II (Chapters 4 to 6) of this text, we mainly concentrated our efforts on electrostatic fields denoted by $\mathbf{E}(x, y, z)$; Part III (Chapters 7 and 8) was devoted to magnetostatic fields represented by $\mathbf{H}(x, y, z)$. We have therefore restricted our discussions to static, or time-invariant, EM fields. Henceforth, we shall examine situations where electric and magnetic fields are dynamic, or time varying. It should be mentioned first that in static EM fields, electric and magnetic fields are independent of each other whereas in dynamic EM fields, the two fields are interdependent. In other words, a time-varying electric field necessarily involves a corresponding time-varying magnetic field. Second, time-varying EM fields, represented by $\mathbf{E}(x, y, z, t)$ and $\mathbf{H}(x, y, z, t)$, are of more practical value than static EM fields. However, familiarity with static fields provides a good background for understanding dynamic fields. Third, recall that electrostatic fields are usually produced by static electric charges whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time-varying fields or waves are usually due to accelerated charges or time-varying currents such as shown in Figure 9.1. Any pulsating current will produce radiation (time-varying fields). It is worth noting that pulsating current of the type shown in Figure 9.1(b) is the cause of radiated emission in digital logic boards. In summary:

stationary charges	→ electrostatic fields
steady currents	→ magnetostatic fields
time-varying currents	→ electromagnetic fields (or waves)

Our aim in this chapter is to lay a firm foundation for our subsequent studies. This will involve introducing two major concepts: (1) electromotive force based on Faraday's experiments, and (2) displacement current, which resulted from Maxwell's hypothesis. As a result of these concepts, Maxwell's equations as presented in Section 7.6 and the boundary

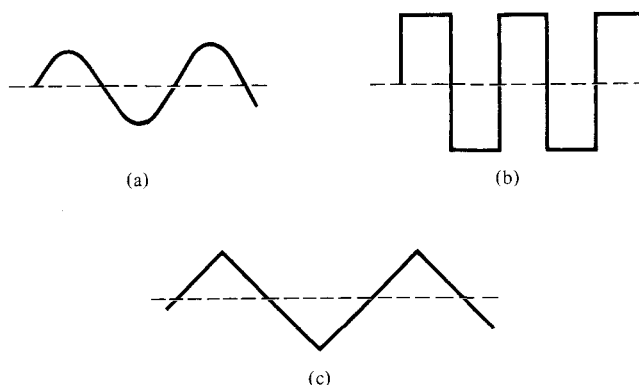


Figure 9.1 Various types of time-varying current: (a) sinusoidal, (b) rectangular, (c) triangular.

conditions for static EM fields will be modified to account for the time variation of the fields. It should be stressed that Maxwell's equations summarize the laws of electromagnetism and shall be the basis of our discussions in the remaining part of the text. For this reason, Section 9.5 should be regarded as the heart of this text.

9.2 FARADAY'S LAW

After Oersted's experimental discovery (upon which Biot–Savart and Ampere based their laws) that a steady current produces a magnetic field, it seemed logical to find out if magnetism would produce electricity. In 1831, about 11 years after Oersted's discovery, Michael Faraday in London and Joseph Henry in New York discovered that a time-varying magnetic field would produce an electric current.¹

According to Faraday's experiments, a static magnetic field produces no current flow, but a time-varying field produces an induced voltage (called *electromotive force* or simply *emf*) in a closed circuit, which causes a flow of current.

Faraday discovered that the **induced emf**, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

This is called *Faraday's law*, and it can be expressed as

$$V_{\text{emf}} = -\frac{d\lambda}{dt} = -N \frac{d\Psi}{dt} \quad (9.1)$$

where N is the number of turns in the circuit and Ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux produc-

¹For details on the experiments of Michael Faraday (1791–1867) and Joseph Henry (1797–1878), see W. F. Magie, *A Source Book in Physics*. Cambridge, MA: Harvard Univ. Press, 1963, pp. 472–519.

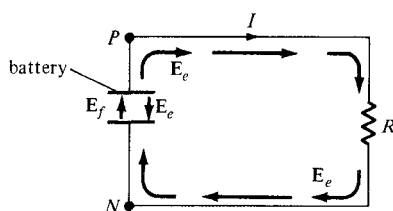


Figure 9.2 A circuit showing emf-producing field $\mathbf{E}_f$ and electrostatic field $\mathbf{E}_e$.

ing it. This is known as *Lenz's law*,² and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the original magnetic field.

Recall that we described an electric field as one in which electric charges experience force. The electric fields considered so far are caused by electric charges; in such fields, the flux lines begin and end on the charges. However, there are other kinds of electric fields not directly caused by electric charges. These are emf-produced fields. Sources of emf include electric generators, batteries, thermocouples, fuel cells, and photovoltaic cells, which all convert nonelectrical energy into electrical energy.

Consider the electric circuit of Figure 9.2, where the battery is a source of emf. The electrochemical action of the battery results in an emf-produced field $\mathbf{E}_f$. Due to the accumulation of charge at the battery terminals, an electrostatic field $\mathbf{E}_e$ ($= -\nabla V$) also exists. The total electric field at any point is

$$\mathbf{E} = \mathbf{E}_f + \mathbf{E}_e \quad (9.2)$$

Note that $\mathbf{E}_f$ is zero outside the battery, $\mathbf{E}_f$ and $\mathbf{E}_e$ have opposite directions in the battery, and the direction of $\mathbf{E}_e$ inside the battery is opposite to that outside it. If we integrate eq. (9.2) over the closed circuit,

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \oint_L \mathbf{E}_f \cdot d\mathbf{l} + 0 = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} \quad (\text{through battery}) \quad (9.3a)$$

where $\oint \mathbf{E}_e \cdot d\mathbf{l} = 0$ because $\mathbf{E}_e$ is conservative. The emf of the battery is the line integral of the emf-produced field; that is,

$$V_{\text{emf}} = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} = - \int_N^P \mathbf{E}_e \cdot d\mathbf{l} = IR \quad (9.3b)$$

since $\mathbf{E}_f$ and $\mathbf{E}_e$ are equal but opposite within the battery (see Figure 9.2). It may also be regarded as the potential difference ($V_P - V_N$) between the battery's open-circuit terminals. It is important to note that:

1. An electrostatic field $\mathbf{E}_e$ cannot maintain a steady current in a closed circuit since $\oint_L \mathbf{E}_e \cdot d\mathbf{l} = 0 = IR$.
2. An emf-produced field $\mathbf{E}_f$ is nonconservative.
3. Except in electrostatics, voltage and potential difference are usually not equivalent.

²After Heinrich Friedrich Emil Lenz (1804–1865), a Russian professor of physics.

9.3 TRANSFORMER AND MOTIONAL EMFs

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single turn ($N = 1$), eq. (9.1) becomes

$$V_{\text{emf}} = -\frac{d\Psi}{dt} \quad (9.4)$$

In terms of $\mathbf{E}$ and $\mathbf{B}$, eq. (9.4) can be written as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad (9.5)$$

where Ψ has been replaced by $\int_S \mathbf{B} \cdot d\mathbf{S}$ and S is the surface area of the circuit bounded by the closed path L . It is clear from eq. (9.5) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that $d\mathbf{l}$ and $d\mathbf{S}$ in eq. (9.5) are in accordance with the right-hand rule as well as Stokes's theorem. This should be observed in Figure 9.3. The variation of flux with time as in eq. (9.1) or eq. (9.5) may be caused in three ways:

1. By having a stationary loop in a time-varying $\mathbf{B}$ field
2. By having a time-varying loop area in a static $\mathbf{B}$ field
3. By having a time-varying loop area in a time-varying $\mathbf{B}$ field.

Each of these will be considered separately.

A. Stationary Loop in Time-Varying $\mathbf{B}$ Field (Transformer emf)

This is the case portrayed in Figure 9.3 where a stationary conducting loop is in a time-varying magnetic $\mathbf{B}$ field. Equation (9.5) becomes

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad (9.6)$$

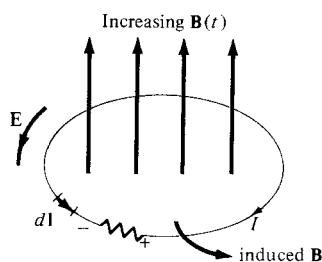


Figure 9.3 Induced emf due to a stationary loop in a time-varying $\mathbf{B}$ field.

This emf induced by the time-varying current (producing the time-varying $\mathbf{B}$ field) in a stationary loop is often referred to as *transformer emf* in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. (9.6), we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad (9.7)$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\boxed{\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}} \quad (9.8)$$

This is one of the Maxwell's equations for time-varying fields. It shows that the time-varying $\mathbf{E}$ field is not conservative ($\nabla \times \mathbf{E} \neq 0$). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time-varying electric field, for example, is due to the energy from the time-varying magnetic field. Observe that Figure 9.3 obeys Lenz's law; the induced current I flows such as to produce a magnetic field that opposes $\mathbf{B}(t)$.

B. Moving Loop in Static $\mathbf{B}$ Field (Motional emf)

When a conducting loop is moving in a static $\mathbf{B}$ field, an emf is induced in the loop. We recall from eq. (8.2) that the force on a charge moving with uniform velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \quad (8.2)$$

We define the *motional electric field* $\mathbf{E}_m$ as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B} \quad (9.9)$$

If we consider a conducting loop, moving with uniform velocity $\mathbf{u}$ as consisting of a large number of free electrons, the emf induced in the loop is

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}} \quad (9.10)$$

This type of emf is called *motional emf* or *flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators. Figure 9.4 illustrates a two-pole dc machine with one armature coil and a two-bar commutator. Although the analysis of the d.c. machine is beyond the scope of this text, we can see that voltage is generated as the coil rotates within the magnetic field. Another example of motional emf is illustrated in Figure 9.5, where a rod is moving between a pair

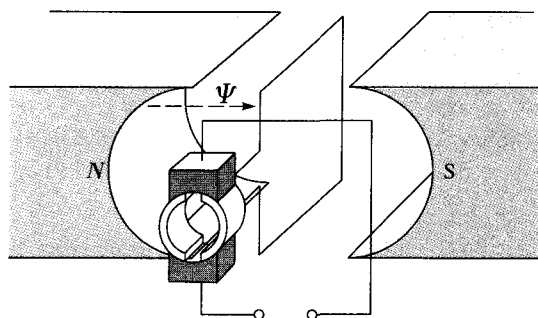


Figure 9.4 A direct-current machine.

of rails. In this example, $\mathbf{B}$ and $\mathbf{u}$ are perpendicular, so eq. (9.9) in conjunction with eq. (8.2) becomes

$$\mathbf{F}_m = I\ell \times \mathbf{B} \quad (9.11)$$

or

$$F_m = I\ell B \quad (9.12)$$

and eq. (9.10) becomes

$$V_{\text{emf}} = uB\ell \quad (9.13)$$

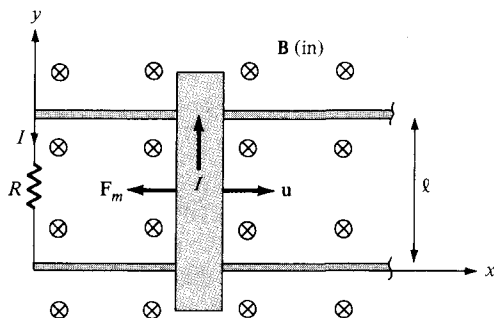
By applying Stokes's theorem to eq. (9.10)

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

or

$$\boxed{\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})} \quad (9.14)$$

Notice that unlike eq. (9.6), there is no need for a negative sign in eq. (9.10) because Lenz's law is already accounted for.

Figure 9.5 Induced emf due to a moving loop in a static $\mathbf{B}$ field.

To apply eq. (9.10) is not always easy; some care must be exercised. The following points should be noted:

1. The integral in eq. (9.10) is zero along the portion of the loop where $\mathbf{u} = 0$. Thus $d\mathbf{l}$ is taken along the portion of the loop that is cutting the field (along the rod in Figure 9.5), where $\mathbf{u}$ has nonzero value.
2. The direction of the induced current is the same as that of $\mathbf{E}_m$ or $\mathbf{u} \times \mathbf{B}$. The limits of the integral in eq. (9.10) are selected in the opposite direction to the induced current thereby satisfying Lenz's law. In eq. (9.13), for example, the integration over L is along $-\mathbf{a}_y$, whereas induced current flows in the rod along $\mathbf{a}_y$.

C. Moving Loop in Time-Varying Field

This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining eqs. (9.6) and (9.10) gives the total emf as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \quad (9.15)$$

or from eqs. (9.8) and (9.14),

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \quad (9.16)$$

Note that eq. (9.15) is equivalent to eq. (9.4), so V_{emf} can be found using either eq. (9.15) or (9.4). In fact, eq. (9.4) can always be applied in place of eqs. (9.6), (9.10), and (9.15).

EXAMPLE 9.1

A conducting bar can slide freely over two conducting rails as shown in Figure 9.6. Calculate the induced voltage in the bar

- (a) If the bar is stationed at $y = 8$ cm and $\mathbf{B} = 4 \cos 10^6 t \mathbf{a}_z$ mWb/m²
- (b) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4\mathbf{a}_z$ mWb/m²
- (c) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4 \cos (10^6 t - y) \mathbf{a}_z$ mWb/m²

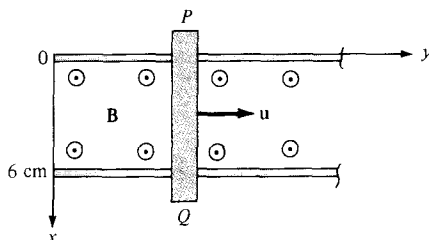


Figure 9.6 For Example 9.1.

Solution:

(a) In this case, we have transformer emf given by

$$\begin{aligned} V_{\text{emf}} &= - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4(10^{-3})(10^6) \sin 10^6 t \, dx \, dy \\ &= 4(10^3)(0.08)(0.06) \sin 10^6 t \\ &= 19.2 \sin 10^6 t \, \text{V} \end{aligned}$$

The polarity of the induced voltage (according to Lenz's law) is such that point P on the bar is at lower potential than Q when $\mathbf{B}$ is increasing.

(b) This is the case of motional emf:

$$\begin{aligned} V_{\text{emf}} &= \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{x=\ell}^0 (u\mathbf{a}_y \times B\mathbf{a}_z) \cdot dx\mathbf{a}_x \\ &= -uB\ell = -20(4 \cdot 10^{-3})(0.06) \\ &= -4.8 \, \text{mV} \end{aligned}$$

(c) Both transformer emf and motional emf are present in this case. This problem can be solved in two ways.

Method 1: Using eq. (9.15)

$$\begin{aligned} V_{\text{emf}} &= - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \quad (9.1.1) \\ &= \int_{x=0}^{0.06} \int_0^y 4 \cdot 10^{-3}(10^6) \sin(10^6 t - y') dy' \, dx \\ &\quad + \int_{0.06}^0 [20\mathbf{a}_y \times 4 \cdot 10^{-3} \cos(10^6 t - y)\mathbf{a}_z] \cdot dx \, \mathbf{a}_x \\ &= 240 \cos(10^6 t - y') \Big|_0^y - 80(10^{-3})(0.06) \cos(10^6 t - y) \\ &= 240 \cos(10^6 t - y) - 240 \cos 10^6 t - 4.8(10^{-3}) \cos(10^6 t - y) \\ &\approx 240 \cos(10^6 t - y) - 240 \cos 10^6 t \quad (9.1.2) \end{aligned}$$

because the motional emf is negligible compared with the transformer emf. Using trigonometric identity

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$V_{\text{emf}} = 480 \sin \left(10^6 t - \frac{y}{2} \right) \sin \frac{y}{2} \, \text{V} \quad (9.1.3)$$

Method 2: Alternatively we can apply eq. (9.4), namely,

$$V_{\text{emf}} = -\frac{\partial \Psi}{\partial t} \quad (9.1.4)$$

where

$$\begin{aligned} \Psi &= \int \mathbf{B} \cdot d\mathbf{S} \\ &= \int_{y=0}^y \int_{x=0}^{0.06} 4 \cos(10^6 t - y) dx dy \\ &= -4(0.06) \sin(10^6 t - y) \Big|_{y=0}^y \\ &= -0.24 \sin(10^6 t - y) + 0.24 \sin 10^6 t \text{ mWb} \end{aligned}$$

But

$$\frac{dy}{dt} = u \rightarrow y = ut = 20t$$

Hence,

$$\begin{aligned} \Psi &= -0.24 \sin(10^6 t - 20t) + 0.24 \sin 10^6 t \text{ mWb} \\ V_{\text{emf}} &= -\frac{\partial \Psi}{\partial t} = 0.24(10^6 - 20) \cos(10^6 t - 20t) - 0.24(10^6) \cos 10^6 t \text{ mV} \\ &\simeq 240 \cos(10^6 t - y) - 240 \cos 10^6 t \text{ V} \end{aligned} \quad (9.1.5)$$

which is the same result in (9.1.2). Notice that in eq. (9.1.1), the dependence of y on time is taken care of in $\int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$, and we should not be bothered by it in $\partial \mathbf{B} / \partial t$. Why? Because the loop is assumed stationary when computing the transformer emf. This is a subtle point one must keep in mind in applying eq. (9.1.1). For the same reason, the second method is always easier.

PRACTICE EXERCISE 9.1

Consider the loop of Figure 9.5. If $\mathbf{B} = 0.5\mathbf{a}_z \text{ Wb/m}^2$, $R = 20 \Omega$, $\ell = 10 \text{ cm}$, and the rod is moving with a constant velocity of $8\mathbf{a}_x \text{ m/s}$, find

- The induced emf in the rod
- The current through the resistor
- The motional force on the rod
- The power dissipated by the resistor.

Answer: (a) 0.4 V, (b) 20 mA, (c) $-\mathbf{a}_x \text{ mN}$, (d) 8 mW.

EXAMPLE 9.2

The loop shown in Figure 9.7 is inside a uniform magnetic field $\mathbf{B} = 50 \mathbf{a}_x$ mWb/m². If side DC of the loop cuts the flux lines at the frequency of 50 Hz and the loop lies in the yz -plane at time $t = 0$, find

- The induced emf at $t = 1$ ms
- The induced current at $t = 3$ ms

Solution:

- Since the $\mathbf{B}$ field is time invariant, the induced emf is motional, that is,

$$V_{\text{emf}} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

where

$$d\mathbf{l} = d\mathbf{l}_{DC} = dz \mathbf{a}_z, \quad \mathbf{u} = \frac{d\mathbf{l}'}{dt} = \frac{\rho d\phi}{dt} \mathbf{a}_\phi = \rho\omega \mathbf{a}_\phi$$

$$\rho = AD = 4 \text{ cm}, \quad \omega = 2\pi f = 100\pi$$

As $\mathbf{u}$ and $d\mathbf{l}$ are in cylindrical coordinates, we transform $\mathbf{B}$ into cylindrical coordinates using eq. (2.9):

$$\mathbf{B} = B_0 \mathbf{a}_x = B_0 (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi)$$

where $B_0 = 0.05$. Hence,

$$\mathbf{u} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\omega & 0 \\ B_0 \cos \phi & -B_0 \sin \phi & 0 \end{vmatrix} = -\rho\omega B_0 \cos \phi \mathbf{a}_z$$

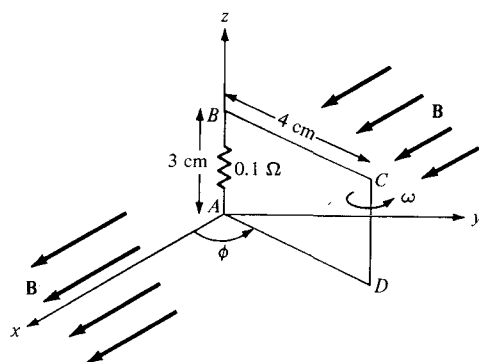


Figure 9.7 For Example 9.2; polarity is for increasing emf.

and

$$(\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\rho\omega B_o \cos \phi \, dz = -0.04(100\pi)(0.05) \cos \phi \, dz \\ = -0.2\pi \cos \phi \, dz$$

$$V_{\text{emf}} = \int_{z=0}^{0.03} -0.2\pi \cos \phi \, dz = -6\pi \cos \phi \, \text{mV}$$

To determine ϕ , recall that

$$\omega = \frac{d\phi}{dt} \rightarrow \phi = \omega t + C_o$$

where C_o is an integration constant. At $t = 0$, $\phi = \pi/2$ because the loop is in the yz -plane at that time, $C_o = \pi/2$. Hence,

$$\phi = \omega t + \frac{\pi}{2}$$

and

$$V_{\text{emf}} = -6\pi \cos\left(\omega t + \frac{\pi}{2}\right) = 6\pi \sin(100\pi t) \, \text{mV}$$

$$\text{At } t = 1 \, \text{ms}, V_{\text{emf}} = 6\pi \sin(0.1\pi) = 5.825 \, \text{mV}$$

(b) The current induced is

$$i = \frac{V_{\text{emf}}}{R} = 60\pi \sin(100\pi t) \, \text{mA}$$

At $t = 3 \, \text{ms}$,

$$i = 60\pi \sin(0.3\pi) \, \text{mA} = 0.1525 \, \text{A}$$

PRACTICE EXERCISE 9.2

Rework Example 9.2 with everything the same except that the $\mathbf{B}$ field is changed to:

- (a) $\mathbf{B} = 50\mathbf{a}_x \, \text{mWb/m}^2$ —that is, the magnetic field is oriented along the y -direction
- (b) $\mathbf{B} = 0.02t \, \mathbf{a}_x \, \text{Wb/m}^2$ —that is, the magnetic field is time varying.

Answer: (a) $-17.93 \, \text{mV}$, $-0.1108 \, \text{A}$, (b) $20.5 \, \mu\text{V}$, $-41.92 \, \text{mA}$.

EXAMPLE 9.3

The magnetic circuit of Figure 9.8 has a uniform cross section of $10^{-3} \, \text{m}^2$. If the circuit is energized by a current $i_1(t) = 3 \sin 100\pi t \, \text{A}$ in the coil of $N_1 = 200$ turns, find the emf induced in the coil of $N_2 = 100$ turns. Assume that $\mu = 500 \, \mu_0$.

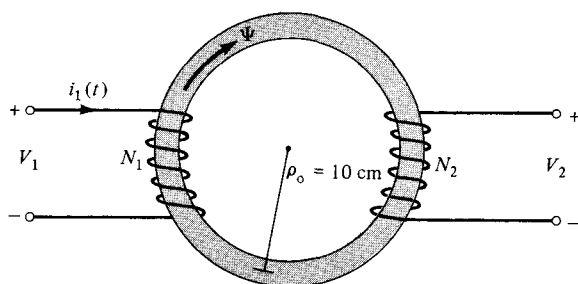


Figure 9.8 Magnetic circuit of Example 9.3.

Solution:

The flux in the circuit is

$$\Psi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{N_1 i_1}{\ell / \mu S} = \frac{N_1 i_1 \mu S}{2\pi \rho_o}$$

According to Faraday's law, the emf induced in the second coil is

$$\begin{aligned} V_2 &= -N_2 \frac{d\Psi}{dt} = -\frac{N_1 N_2 \mu S}{2\pi \rho_o} \frac{di_1}{dt} \\ &= -\frac{100 \cdot (200) \cdot (500) \cdot (4\pi \times 10^{-7}) \cdot (10^{-3}) \cdot 300\pi \cos 100\pi t}{2\pi \cdot (10 \times 10^{-2})} \\ &= -6\pi \cos 100\pi t \text{ V} \end{aligned}$$

PRACTICE EXERCISE 9.3

A magnetic core of uniform cross section 4 cm^2 is connected to a 120-V, 60-Hz generator as shown in Figure 9.9. Calculate the induced emf V_2 in the secondary coil.

Answer: 72 V

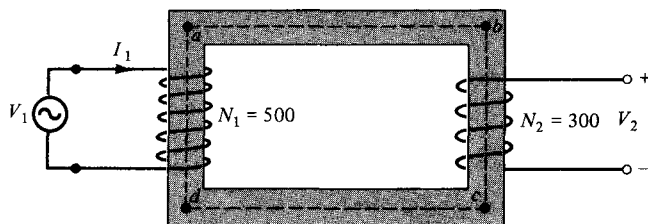


Figure 9.9 For Practice Exercise 9.3.

9.4 DISPLACEMENT CURRENT

In the previous section, we have essentially reconsidered Maxwell's curl equation for electrostatic fields and modified it for time-varying situations to satisfy Faraday's law. We shall now reconsider Maxwell's curl equation for magnetic fields (Ampere's circuit law) for time-varying conditions.

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (9.17)$$

But the divergence of the curl of any vector field is identically zero (see Example 3.10). Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \quad (9.18)$$

The continuity of current in eq. (5.43), however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \quad (9.19)$$

Thus eqs. (9.18) and (9.19) are obviously incompatible for time-varying conditions. We must modify eq. (9.17) to agree with eq. (9.19). To do this, we add a term to eq. (9.17) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \quad (9.20)$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \quad (9.21)$$

In order for eq. (9.21) to agree with eq. (9.19),

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \quad (9.22a)$$

or

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}} \quad (9.22b)$$

Substituting eq. (9.22b) into eq. (9.20) results in

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \quad (9.23)$$

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D} / \partial t$ is known as *displacement current density* and $\mathbf{J}$ is the conduction current

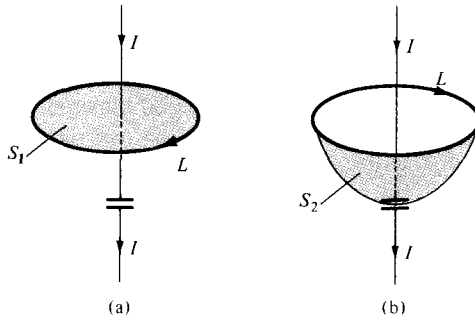


Figure 9.10 Two surfaces of integration showing the need for $\mathbf{J}_d$ in Ampere's circuit law.

density ($\mathbf{J} = \sigma \mathbf{E}$).³ The insertion of $\mathbf{J}_d$ into eq. (9.17) was one of the major contributions of Maxwell. Without the term $\mathbf{J}_d$, electromagnetic wave propagation (radio or TV waves, for example) would be impossible. At low frequencies, $\mathbf{J}_d$ is usually neglected compared with $\mathbf{J}$. However, at radio frequencies, the two terms are comparable. At the time of Maxwell, high-frequency sources were not available and eq. (9.23) could not be verified experimentally. It was years later that Hertz succeeded in generating and detecting radio waves thereby verifying eq. (9.23). This is one of the rare situations where mathematical argument paved the way for experimental investigation.

Based on the displacement current density, we define the *displacement current* as

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{S} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \quad (9.24)$$

We must bear in mind that displacement current is a result of time-varying electric field. A typical example of such current is the current through a capacitor when an alternating voltage source is applied to its plates. This example, shown in Figure 9.10, serves to illustrate the need for the displacement current. Applying an unmodified form of Ampere's circuit law to a closed path L shown in Figure 9.10(a) gives

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = I \quad (9.25)$$

where I is the current through the conductor and S_1 is the flat surface bounded by L . If we use the balloon-shaped surface S_2 that passes between the capacitor plates, as in Figure 9.10(b),

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = 0 \quad (9.26)$$

because no conduction current ($\mathbf{J} = 0$) flows through S_2 . This is contradictory in view of the fact that the same closed path L is used. To resolve the conflict, we need to include the

³Recall that we also have $\mathbf{J} = \rho_v \mathbf{u}$ as the convection current density.

displacement current in Ampere's circuit law. The total current density is $\mathbf{J} + \mathbf{J}_d$. In eq. (9.25), $\mathbf{J}_d = 0$ so that the equation remains valid. In eq. (9.26), $\mathbf{J} = 0$ so that

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J}_d \cdot d\mathbf{S} = \frac{d}{dt} \int_{S_2} \mathbf{D} \cdot d\mathbf{S} = \frac{dQ}{dt} = I \quad (9.27)$$

So we obtain the same current for either surface though it is conduction current in S_1 and displacement current in S_2 .

EXAMPLE 9.4

A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage $50 \sin 10^3 t \text{ V}$ applied to its plates. Calculate the displacement current assuming $\epsilon = 2\epsilon_0$.

Solution:

$$D = \epsilon E = \epsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\epsilon}{d} \frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot S = \frac{\epsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

which is the same as the conduction current, given by

$$I_c = \frac{dQ}{dt} = S \frac{d\rho_s}{dt} = S \frac{dD}{dt} = \epsilon S \frac{dE}{dt} = \frac{\epsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

$$\begin{aligned} I_d &= 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot 10^3 \times 50 \cos 10^3 t \\ &= 147.4 \cos 10^3 t \text{ nA} \end{aligned}$$

PRACTICE EXERCISE 9.4

In free space, $\mathbf{E} = 20 \cos(\omega t - 50x) \mathbf{a}_y \text{ V/m}$. Calculate

- (a) $\mathbf{J}_d$
- (b) $\mathbf{H}$
- (c) ω

Answer: (a) $-20\omega\epsilon_0 \sin(\omega t - 50x) \mathbf{a}_y \text{ A/m}^2$, (b) $0.4\omega\epsilon_0 \cos(\omega t - 50x) \mathbf{a}_z \text{ A/m}$, (c) $1.5 \times 10^{10} \text{ rad/s}$.

9.5 MAXWELL'S EQUATIONS IN FINAL FORMS

James Clerk Maxwell (1831–1879) is regarded as the founder of electromagnetic theory in its present form. Maxwell's celebrated work led to the discovery of electromagnetic waves.⁴ Through his theoretical efforts over about 5 years (when he was between 35 and 40), Maxwell published the first unified theory of electricity and magnetism. The theory comprised all previously known results, both experimental and theoretical, on electricity and magnetism. It further introduced displacement current and predicted the existence of electromagnetic waves. Maxwell's equations were not fully accepted by many scientists until they were later confirmed by Heinrich Rudolf Hertz (1857–1894), a German physics professor. Hertz was successful in generating and detecting radio waves.

The laws of electromagnetism that Maxwell put together in the form of four equations were presented in Table 7.2 in Section 7.6 for static conditions. The more generalized forms of these equations are those for time-varying conditions shown in Table 9.1. We notice from the table that the divergence equations remain the same while the curl equations have been modified. The integral form of Maxwell's equations depicts the underlying physical laws, whereas the differential form is used more frequently in solving problems. For a field to be "qualified" as an electromagnetic field, it must satisfy all four Maxwell's equations. The importance of Maxwell's equations cannot be overemphasized because they summarize all known laws of electromagnetism. We shall often refer to them in the remaining part of this text.

Since this section is meant to be a compendium of our discussion in this text, it is worthwhile to mention other equations that go hand in hand with Maxwell's equations. The Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (9.28)$$

TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law

*This is also referred to as Gauss's law for magnetic fields.

⁴The work of James Clerk Maxwell (1831–1879), a Scottish physicist, can be found in his book, *A Treatise on Electricity and Magnetism*. New York: Dover, vols. 1 and 2, 1954.

is associated with Maxwell's equations. Also the equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \quad (9.29)$$

is implicit in Maxwell's equations. The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time-varying fields; in a linear, homogeneous, and isotropic medium characterized by σ , ϵ , and μ , the constitutive relations

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (9.30a)$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 (\mathbf{H} + \mathbf{M}) \quad (9.30b)$$

$$\mathbf{J} = \sigma \mathbf{E} + \rho_v \mathbf{u} \quad (9.30c)$$

hold for time-varying fields. Consequently, the boundary conditions

$$E_{1t} = E_{2t} \quad \text{or} \quad (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_{n12} = 0 \quad (9.31a)$$

$$H_{1t} - H_{2t} = K \quad \text{or} \quad (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \quad (9.31b)$$

$$D_{1n} - D_{2n} = \rho_s \quad \text{or} \quad (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_{n12} = \rho_s \quad (9.31c)$$

$$B_{1n} - B_{2n} = 0 \quad \text{or} \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{a}_{n12} = 0 \quad (9.31d)$$

remain valid for time-varying fields. However, for a perfect conductor ($\sigma \approx \infty$) in a time-varying field,

$$\mathbf{E} = 0, \quad \mathbf{H} = 0, \quad \mathbf{J} = 0 \quad (9.32)$$

and hence,

$$\mathbf{B}_n = 0, \quad \mathbf{E}_t = 0 \quad (9.33)$$

For a perfect dielectric ($\sigma \approx 0$), eq. (9.31) holds except that $\mathbf{K} = 0$. Though eqs. (9.28) to (9.33) are not Maxwell's equations, they are associated with them.

To complete this summary section, we present a structure linking the various potentials and vector fields of the electric and magnetic fields in Figure 9.11. This electromagnetic flow diagram helps with the visualization of the basic relationships between field quantities. It also shows that it is usually possible to find alternative formulations, for a given problem, in a relatively simple manner. It should be noted that in Figures 9.10(b) and (c), we introduce ρ^m as the free magnetic density (similar to ρ_v), which is, of course, zero, $\mathbf{A}_e$ as the magnetic current density (analogous to $\mathbf{J}$). Using terms from stress analysis, the principal relationships are typified as:

(a) compatibility equations

$$\nabla \cdot \mathbf{B} = \rho^m = 0 \quad (9.34)$$

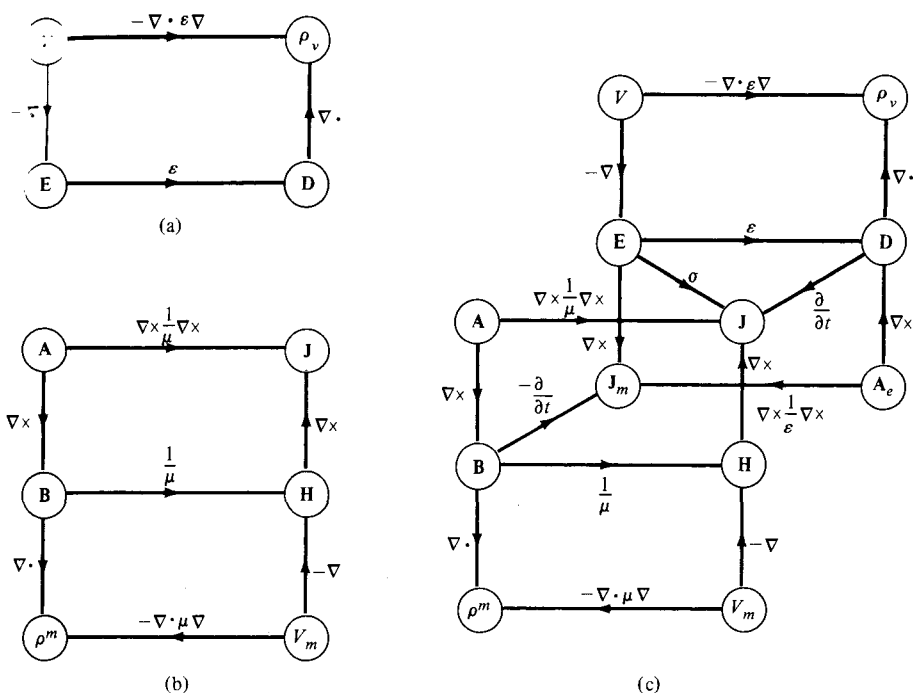


Figure 9.11 Electromagnetic flow diagram showing the relationship between the potentials and vector fields: (a) electrostatic system, (b) magnetostatic system, (c) electromagnetic system. [Adapted with permission from IEE Publishing Dept.]

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \mathbf{J}_m \quad (9.35)$$

(b) constitutive equations

$$\mathbf{B} = \mu \mathbf{H} \quad (9.36)$$

and

$$\mathbf{D} = \epsilon \mathbf{E} \quad (9.37)$$

(c) equilibrium equations

$$\nabla \cdot \mathbf{D} = \rho_v \quad (9.38)$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (9.39)$$

9.6 TIME-VARYING POTENTIALS

For static EM fields, we obtained the electric scalar potential as

$$V = \int_v \frac{\rho_v dv}{4\pi\epsilon R} \quad (9.40)$$

and the magnetic vector potential as

$$\mathbf{A} = \int_v \frac{\mu \mathbf{J} dv}{4\pi R} \quad (9.41)$$

We would like to examine what happens to these potentials when the fields are time varying. Recall that $\mathbf{A}$ was defined from the fact that $\nabla \cdot \mathbf{B} = 0$, which still holds for time-varying fields. Hence the relation

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \quad (9.42)$$

holds for time-varying situations. Combining Faraday's law in eq. (9.8) with eq. (9.42) gives

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A}) \quad (9.43a)$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (9.43b)$$

Since the curl of the gradient of a scalar field is identically zero (see Practice Exercise 3.10), the solution to eq. (9.43b) is

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \quad (9.44)$$

or

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \quad (9.45)$$

From eqs. (9.42) and (9.45), we can determine the vector fields $\mathbf{B}$ and $\mathbf{E}$ provided that the potentials $\mathbf{A}$ and V are known. However, we still need to find some expressions for $\mathbf{A}$ and V similar to those in eqs. (9.40) and (9.41) that are suitable for time-varying fields.

From Table 9.1 or eq. (9.38) we know that $\nabla \cdot \mathbf{D} = \rho_v$ is valid for time-varying conditions. By taking the divergence of eq. (9.45) and making use of eqs. (9.37) and (9.38), we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon} = -\nabla^2 V - \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A})$$

or

$$\nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho_v}{\epsilon} \quad (9.46)$$

Taking the curl of eq. (9.42) and incorporating eqs. (9.23) and (9.45) results in

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A} &= \mu \mathbf{J} + \epsilon \mu \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) \\ &= \mu \mathbf{J} - \mu \epsilon \nabla \left(\frac{\partial V}{\partial t} \right) - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \end{aligned} \quad (9.47)$$

where $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ have been assumed. By applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (9.48)$$

to eq. (9.47),

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mu \mathbf{J} + \mu \epsilon \nabla \left(\frac{\partial V}{\partial t} \right) + \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \quad (9.49)$$

A vector field is uniquely defined when its curl and divergence are specified. The curl of $\mathbf{A}$ has been specified by eq. (9.42); for reasons that will be obvious shortly, we may choose the divergence of $\mathbf{A}$ as

$$\boxed{\nabla \cdot \mathbf{A} = -\mu \epsilon \frac{\partial V}{\partial t}} \quad (9.50)$$

This choice relates $\mathbf{A}$ and V and it is called the *Lorentz condition for potentials*. We had this in mind when we chose $\nabla \cdot \mathbf{A} = 0$ for magnetostatic fields in eq. (7.59). By imposing the Lorentz condition of eq. (9.50), eqs. (9.46) and (9.49), respectively, become

$$\boxed{\nabla^2 V - \mu \epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\epsilon}} \quad (9.51)$$

and

$$\boxed{\nabla^2 \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}} \quad (9.52)$$

which are *wave equations* to be discussed in the next chapter. The reason for choosing the Lorentz condition becomes obvious as we examine eqs. (9.51) and (9.52). It uncouples eqs. (9.46) and (9.49) and also produces a symmetry between eqs. (9.51) and (9.52). It can be shown that the Lorentz condition can be obtained from the continuity equation; therefore, our choice of eq. (9.50) is not arbitrary. Notice that eqs. (6.4) and (7.60) are special static cases of eqs. (9.51) and (9.52), respectively. In other words, potentials V and $\mathbf{A}$ satisfy Poisson's equations for time-varying conditions. Just as eqs. (9.40) and (9.41) are

the solutions, or the integral forms of eqs. (6.4) and (7.60), it can be shown that the solutions⁵ to eqs. (9.51) and (9.52) are

$$V = \int_v \frac{[\rho_v] dv}{4\pi\epsilon R} \quad (9.53)$$

and

$$\mathbf{A} = \int_v \frac{\mu[\mathbf{J}] dv}{4\pi R} \quad (9.54)$$

The term $[\rho_v]$ (or $[\mathbf{J}]$) means that the time t in $\rho_v(x, y, z, t)$ [or $\mathbf{J}(x, y, z, t)$] is replaced by the *retarded time* t' given by

$$t' = t - \frac{R}{u} \quad (9.55)$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the source point $\mathbf{r}'$ and the observation point $\mathbf{r}$ and

$$u = \frac{1}{\sqrt{\mu\epsilon}} \quad (9.56)$$

is the velocity of wave propagation. In free space, $u = c \approx 3 \times 10^8$ m/s is the speed of light in a vacuum. Potentials V and $\mathbf{A}$ in eqs. (9.53) and (9.54) are, respectively, called the *retarded electric scalar potential* and the *retarded magnetic vector potential*. Given ρ_v and $\mathbf{J}$, V and $\mathbf{A}$ can be determined using eqs. (9.53) and (9.54); from V and $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ can be determined using eqs. (9.45) and (9.42), respectively.

9.7 TIME-HARMONIC FIELDS

So far, our time dependence of EM fields has been arbitrary. To be specific, we shall assume that the fields are *time harmonic*.

A time-harmonic field is one that varies periodically or sinusoidally with time.

Not only is sinusoidal analysis of practical value, it can be extended to most waveforms by Fourier transform techniques. Sinusoids are easily expressed in phasors, which are more convenient to work with. Before applying phasors to EM fields, it is worthwhile to have a brief review of the concept of phasor.

A *phasor* z is a complex number that can be written as

$$z = x + jy = r \angle \phi \quad (9.57)$$

⁵For example, see D. K. Cheng, *Field and Wave Electromagnetics*, Reading, MA: Addison-Wesley, 1983, pp. 291–292.

or

$$z = r e^{j\phi} = r (\cos \phi + j \sin \phi) \quad (9.58)$$

where $j = \sqrt{-1}$, x is the real part of z , y is the imaginary part of z , r is the magnitude of z , given by

$$r = |z| = \sqrt{x^2 + y^2} \quad (9.59)$$

and ϕ is the phase of z , given by

$$\phi = \tan^{-1} \frac{y}{x} \quad (9.60)$$

Here x , y , z , r , and ϕ should not be mistaken as the coordinate variables although they look similar (different letters could have been used but it is hard to find better ones). The phasor z can be represented in *rectangular form* as $z = x + jy$ or in *polar form* as $z = r \angle \phi = r e^{j\phi}$. The two forms of representing z are related in eqs. (9.57) to (9.60) and illustrated in Figure 9.12. Addition and subtraction of phasors are better performed in rectangular form; multiplication and division are better done in polar form.

Given complex numbers

$$z = x + jy = r \angle \phi, \quad z_1 = x_1 + jy_1 = r_1 \angle \phi_1, \quad \text{and} \quad z_2 = x_2 + jy_2 = r_2 \angle \phi_2$$

the following basic properties should be noted.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \quad (9.61a)$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \quad (9.61b)$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2 \quad (9.61c)$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_2 \quad (9.61d)$$

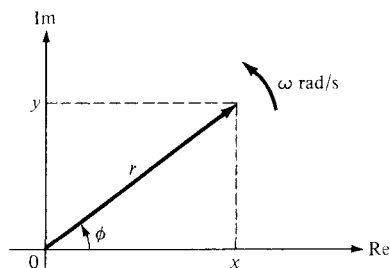


Figure 9.12 Representation of a phasor $z = x + jy = r \angle \phi$.

Square Root:

$$\sqrt{z} = \sqrt{r} \angle \phi/2 \quad (9.61e)$$

Complex Conjugate:

$$z^* = x - jy = r \angle -\phi = re^{-j\phi} \quad (9.61f)$$

Other properties of complex numbers can be found in Appendix A.2.

To introduce the time element, we let

$$\phi = \omega t + \theta \quad (9.62)$$

where θ may be a function of time or space coordinates or a constant. The real (Re) and imaginary (Im) parts of

$$re^{j\phi} = re^{j\theta} e^{j\omega t} \quad (9.63)$$

are, respectively, given by

$$\text{Re}(re^{j\phi}) = r \cos(\omega t + \theta) \quad (9.64a)$$

and

$$\text{Im}(re^{j\phi}) = r \sin(\omega t + \theta) \quad (9.64b)$$

Thus a sinusoidal current $I(t) = I_o \cos(\omega t + \theta)$, for example, equals the real part of $I_o e^{j\theta} e^{j\omega t}$. The current $I'(t) = I_o \sin(\omega t + \theta)$, which is the imaginary part of $I_o e^{j\theta} e^{j\omega t}$, can also be represented as the real part of $I_o e^{j\theta} e^{j\omega t} e^{-j90^\circ}$ because $\sin \alpha = \cos(\alpha - 90^\circ)$. However, in performing our mathematical operations, we must be consistent in our use of either the real part or the imaginary part of a quantity but not both at the same time.

The complex term $I_o e^{j\theta}$, which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called the *phasor* current, denoted by I_s ; that is,

$$I_s = I_o e^{j\theta} = I_o \angle \theta \quad (9.65)$$

where the subscript s denotes the phasor form of $I(t)$. Thus $I(t) = I_o \cos(\omega t + \theta)$, the *instantaneous form*, can be expressed as

$$I(t) = \text{Re}(I_s e^{j\omega t}) \quad (9.66)$$

In general, a phasor could be scalar or vector. If a vector $\mathbf{A}(x, y, z, t)$ is a time-harmonic field, the *phasor form* of $\mathbf{A}$ is $\mathbf{A}_s(x, y, z)$; the two quantities are related as

$$\boxed{\mathbf{A} = \text{Re}(\mathbf{A}_s e^{j\omega t})} \quad (9.67)$$

For example, if $\mathbf{A} = A_o \cos(\omega t - \beta x) \mathbf{a}_y$, we can write $\mathbf{A}$ as

$$\mathbf{A} = \text{Re}(A_o e^{-j\beta x} \mathbf{a}_y e^{j\omega t}) \quad (9.68)$$

Comparing this with eq. (9.67) indicates that the phasor form of $\mathbf{A}$ is

$$\mathbf{A}_s = A_o e^{-j\beta x} \mathbf{a}_y \quad (9.69)$$

Notice from eq. (9.67) that

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{\partial}{\partial t} \operatorname{Re} (\mathbf{A}_s e^{j\omega t}) \\ &= \operatorname{Re} (j\omega \mathbf{A}_s e^{j\omega t})\end{aligned}\quad (9.70)$$

showing that taking the time derivative of the instantaneous quantity is equivalent to multiplying its phasor form by $j\omega$. That is,

$$\frac{\partial \mathbf{A}}{\partial t} \rightarrow j\omega \mathbf{A}_s \quad (9.71)$$

Similarly,

$$\int \mathbf{A} \, dt \rightarrow \frac{\mathbf{A}_s}{j\omega} \quad (9.72)$$

Note that the real part is chosen in eq. (9.67) as in circuit analysis; the imaginary part could equally have been chosen. Also notice the basic difference between the instantaneous form $\mathbf{A}(x, y, z, t)$ and its phasor form $\mathbf{A}_s(x, y, z)$; the former is time dependent and real whereas the latter is time invariant and generally complex. It is easier to work with $\mathbf{A}_s$ and obtain $\mathbf{A}$ from $\mathbf{A}_s$ whenever necessary using eq. (9.67).

We shall now apply the phasor concept to time-varying EM fields. The fields quantities $\mathbf{E}(x, y, z, t)$, $\mathbf{D}(x, y, z, t)$, $\mathbf{H}(x, y, z, t)$, $\mathbf{B}(x, y, z, t)$, $\mathbf{J}(x, y, z, t)$, and $\rho_v(x, y, z, t)$ and their derivatives can be expressed in phasor form using eqs. (9.67) and (9.71). In phasor form, Maxwell's equations for time-harmonic EM fields in a linear, isotropic, and homogeneous medium are presented in Table 9.2. From Table 9.2, note that the time factor $e^{j\omega t}$ disappears because it is associated with every term and therefore factors out, resulting in time-independent equations. Herein lies the justification for using phasors; the time factor can be suppressed in our analysis of time-harmonic fields and inserted when necessary. Also note that in Table 9.2, the time factor $e^{j\omega t}$ has been assumed. It is equally possible to have assumed the time factor $e^{-j\omega t}$, in which case we would need to replace every j in Table 9.2 with $-j$.

TABLE 9.2 Time-Harmonic Maxwell's Equations
Assuming Time Factor $e^{j\omega t}$

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs} \, dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$

EXAMPLE 9.5

Evaluate the complex numbers

$$(a) \ z_1 = \frac{j(3 - j4)^*}{(-1 + j6)(2 + j)^2}$$

$$(b) \ z_2 = \left[\frac{1 + j}{4 - j8} \right]^{1/2}$$

Solution:(a) This can be solved in two ways: working with z in rectangular form or polar form.**Method 1:** (working in rectangular form):

Let

$$z_1 = \frac{z_3 z_4}{z_5 z_6}$$

where

$$z_3 = j$$

$$\begin{aligned} z_4 &= (3 - j4)^* = \text{the complex conjugate of } (3 - j4) \\ &= 3 + j4 \end{aligned}$$

(To find the complex conjugate of a complex number, simply replace every j with $-j$.)

$$z_5 = -1 + j6$$

and

$$z_6 = (2 + j)^2 = 4 - 1 + j4 = 3 + j4$$

Hence,

$$z_3 z_4 = j(3 + j4) = -4 + j3$$

$$\begin{aligned} z_5 z_6 &= (-1 + j6)(3 + j4) = -3 - j4 + j18 - 24 \\ &= -27 + j14 \end{aligned}$$

and

$$z_1 = \frac{-4 + j3}{-27 + j14}$$

Multiplying and dividing z_1 by $-27 - j14$ (rationalization), we have

$$\begin{aligned} z_1 &= \frac{(-4 + j3)(-27 - j14)}{(-27 + j14)(-27 - j14)} = \frac{150 - j25}{27^2 + 14^2} \\ &= 0.1622 - j0.027 = 0.1644 \angle -9.46^\circ \end{aligned}$$

Method 2: (working in polar form):

$$z_3 = j = 1 \angle 90^\circ$$

$$z_4 = (3 - j4)^* = 5 \angle -53.13^\circ = 5 \angle 53.13^\circ$$

$$z_5 = (-1 + j6) = \sqrt{37} \angle 99.46^\circ$$

$$z_6 = (2 + j)^2 = (\sqrt{5} \angle 26.56^\circ)^2 = 5 \angle 53.13^\circ$$

Hence,

$$\begin{aligned} z_1 &= \frac{(1 \angle 90^\circ)(5 \angle 53.13^\circ)}{(\sqrt{37} \angle 99.46^\circ)(5 \angle 53.13^\circ)} \\ &= \frac{1}{\sqrt{37}} \angle 90^\circ - 99.46^\circ = 0.1644 \angle -9.46^\circ \\ &= 0.1622 - j0.027 \end{aligned}$$

as obtained before.

(b) Let

$$z_2 = \left[\frac{z_7}{z_8} \right]^{1/2}$$

where

$$z_7 = 1 + j = \sqrt{2} \angle 45^\circ$$

and

$$z_8 = 4 - j8 = 4\sqrt{5} \angle -63.4^\circ$$

Hence

$$\begin{aligned} \frac{z_7}{z_8} &= \frac{\sqrt{2} \angle 45^\circ}{4\sqrt{5} \angle -63.4^\circ} = \frac{\sqrt{2}}{4\sqrt{5}} \angle 45^\circ - -63.4^\circ \\ &= 0.1581 \angle 108.4^\circ \end{aligned}$$

and

$$\begin{aligned} z_2 &= \sqrt{0.1581} \angle 108.4^\circ / 2 \\ &= 0.3976 \angle 54.2^\circ \end{aligned}$$

PRACTICE EXERCISE 9.5

Evaluate these complex numbers:

(a) $j^3 \left[\frac{1+j}{2-j} \right]^2$

(b) $6 \angle 30^\circ + j5 - 3 + e^{j45^\circ}$

Answer: (a) $0.24 + j0.32$, (b) $2.903 + j8.707$.

EXAMPLE 9.6

Given that $\mathbf{A} = 10 \cos(10^8 t - 10x + 60^\circ) \mathbf{a}_z$ and $\mathbf{B}_s = (20/j) \mathbf{a}_x + 10 e^{j2\pi x/3} \mathbf{a}_y$, express $\mathbf{A}$ in phasor form and $\mathbf{B}_s$ in instantaneous form.

Solution:

$$\mathbf{A} = \text{Re} [10e^{j(\omega t - 10x + 60^\circ)} \mathbf{a}_z]$$

where $\omega = 10^8$. Hence

$$\mathbf{A} = \text{Re} [10e^{j(60^\circ - 10x)} \mathbf{a}_z e^{j\omega t}] = \text{Re} (\mathbf{A}_s e^{j\omega t})$$

or

$$\mathbf{A}_s = 10 e^{j(60^\circ - 10x)} \mathbf{a}_z$$

If

$$\begin{aligned} \mathbf{B}_s &= \frac{20}{j} \mathbf{a}_x + 10 e^{j2\pi x/3} \mathbf{a}_y = -j20 \mathbf{a}_x + 10 e^{j2\pi x/3} \mathbf{a}_y \\ &= 20 e^{-j\pi/2} \mathbf{a}_x + 10 e^{j2\pi x/3} \mathbf{a}_y \end{aligned}$$

$$\begin{aligned} \mathbf{B} &= \text{Re} (\mathbf{B}_s e^{j\omega t}) \\ &= \text{Re} [20 e^{j(\omega t - \pi/2)} \mathbf{a}_x + 10 e^{j(\omega t + 2\pi x/3)} \mathbf{a}_y] \\ &= 20 \cos(\omega t - \pi/2) \mathbf{a}_x + 10 \cos\left(\omega t + \frac{2\pi x}{3}\right) \mathbf{a}_y \\ &= 20 \sin \omega t \mathbf{a}_x + 10 \cos\left(\omega t + \frac{2\pi x}{3}\right) \mathbf{a}_y \end{aligned}$$

PRACTICE EXERCISE 9.6

If $\mathbf{P} = 2 \sin(10t + x - \pi/4) \mathbf{a}_y$ and $\mathbf{Q}_s = e^{jx}(\mathbf{a}_x - \mathbf{a}_z) \sin \pi y$, determine the phasor form of $\mathbf{P}$ and the instantaneous form of $\mathbf{Q}_s$.

Answer: $2e^{j(x - 3\pi/4)} \mathbf{a}_y$, $\sin \pi y \cos(\omega t + x)(\mathbf{a}_x - \mathbf{a}_z)$.

EXAMPLE 9.7

The electric field and magnetic field in free space are given by

$$\mathbf{E} = \frac{50}{\rho} \cos(10^6 t + \beta z) \mathbf{a}_\phi \text{ V/m}$$

$$\mathbf{H} = \frac{H_0}{\rho} \cos(10^6 t + \beta z) \mathbf{a}_\rho \text{ A/m}$$

Express these in phasor form and determine the constants H_0 and β such that the fields satisfy Maxwell's equations.

Solution:

The instantaneous forms of $\mathbf{E}$ and $\mathbf{H}$ are written as

$$\mathbf{E} = \text{Re} (\mathbf{E}_s e^{j\omega t}), \quad \mathbf{H} = \text{Re} (\mathbf{H}_s e^{j\omega t}) \quad (9.7.1)$$

where $\omega = 10^6$ and phasors $\mathbf{E}_s$ and $\mathbf{H}_s$ are given by

$$\mathbf{E}_s = \frac{50}{\rho} e^{j\beta z} \mathbf{a}_\phi, \quad \mathbf{H}_s = \frac{H_o}{\rho} e^{j\beta z} \mathbf{a}_\rho \quad (9.7.2)$$

For free space, $\rho_v = 0$, $\sigma = 0$, $\epsilon = \epsilon_o$, and $\mu = \mu_o$ so Maxwell's equations become

$$\nabla \cdot \mathbf{D} = \epsilon_o \nabla \cdot \mathbf{E} = 0 \rightarrow \nabla \cdot \mathbf{E}_s = 0 \quad (9.7.3)$$

$$\nabla \cdot \mathbf{B} = \mu_o \nabla \cdot \mathbf{H} = 0 \rightarrow \nabla \cdot \mathbf{H}_s = 0 \quad (9.7.4)$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \rightarrow \nabla \times \mathbf{H}_s = j\omega \epsilon_o \mathbf{E}_s \quad (9.7.5)$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \rightarrow \nabla \times \mathbf{E}_s = -j\omega \mu_o \mathbf{H}_s \quad (9.7.6)$$

Substituting eq. (9.7.2) into eqs. (9.7.3) and (9.7.4), it is readily verified that two Maxwell's equations are satisfied; that is,

$$\nabla \cdot \mathbf{E}_s = \frac{1}{\rho} \frac{\partial}{\partial \phi} (E_{\phi s}) = 0$$

$$\nabla \cdot \mathbf{H}_s = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_{\rho s}) = 0$$

Now

$$\nabla \times \mathbf{H}_s = \nabla \times \left(\frac{H_o}{\rho} e^{j\beta z} \mathbf{a}_\rho \right) = \frac{jH_o \beta}{\rho} e^{j\beta z} \mathbf{a}_\phi \quad (9.7.7)$$

Substituting eqs. (9.7.2) and (9.7.7) into eq. (9.7.5), we have

$$\frac{jH_o \beta}{\rho} e^{j\beta z} \mathbf{a}_\phi = j\omega \epsilon_o \frac{50}{\rho} e^{j\beta z} \mathbf{a}_\phi$$

or

$$H_o \beta = 50 \omega \epsilon_o \quad (9.7.8)$$

Similarly, substituting eq. (9.7.2) into (9.7.6) gives

$$-j\beta \frac{50}{\rho} e^{j\beta z} \mathbf{a}_\rho = -j\omega \mu_o \frac{H_o}{\rho} e^{j\beta z} \mathbf{a}_\rho$$

or

$$\frac{H_o}{\beta} = \frac{50}{\omega \mu_o} \quad (9.7.9)$$

Multiplying eq. (9.7.8) with eq. (9.7.9) yields

$$H_o^2 = (50)^2 \frac{\epsilon_o}{\mu_o}$$

or

$$H_o = \pm 50 \sqrt{\epsilon_o / \mu_o} = \pm \frac{50}{120\pi} = \pm 0.1326$$

Dividing eq. (9.7.8) by eq. (9.7.9), we get

$$\beta^2 = \omega^2 \mu_o \epsilon_o$$

or

$$\begin{aligned} \beta &= \pm \omega \sqrt{\mu_o \epsilon_o} = \pm \frac{\omega}{c} = \pm \frac{10^6}{3 \times 10^8} \\ &= \pm 3.33 \times 10^{-3} \end{aligned}$$

In view of eq. (9.7.8), $H_o = 0.1326$, $\beta = 3.33 \times 10^{-3}$ or $H_o = -0.1326$, $\beta = -3.33 \times 10^{-3}$; only these will satisfy Maxwell's four equations.

PRACTICE EXERCISE 9.7

In air, $\mathbf{E} = \frac{\sin \theta}{r} \cos(6 \times 10^7 t - \beta r) \mathbf{a}_\phi$ V/m.

Find β and $\mathbf{H}$.

Answer: 0.2 rad/m, $-\frac{1}{12\pi r^2} \cos \theta \sin(6 \times 10^7 t - 0.2r) \mathbf{a}_r - \frac{1}{120\pi r} \sin \theta \times \cos(6 \times 10^7 t - 0.2r) \mathbf{a}_\theta$ A/m.

EXAMPLE 9.8

In a medium characterized by $\sigma = 0$, $\mu = \mu_o$, ϵ_o , and

$$\mathbf{E} = 20 \sin(10^8 t - \beta z) \mathbf{a}_y \text{ V/m}$$

calculate β and $\mathbf{H}$.

Solution:

This problem can be solved directly in time domain or using phasors. As in the previous example, we find β and $\mathbf{H}$ by making $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's four equations.

Method 1 (time domain): Let us solve this problem the harder way—in time domain. It is evident that Gauss's law for electric fields is satisfied; that is,

$$\nabla \cdot \mathbf{E} = \frac{\partial E_y}{\partial y} = 0$$

From Faraday's law,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \rightarrow \quad \mathbf{H} = -\frac{1}{\mu} \int (\nabla \times \mathbf{E}) dt$$

But

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z} \mathbf{a}_x + \frac{\partial E_y}{\partial x} \mathbf{a}_z \\ &= 20\beta \cos(10^8 t - \beta z) \mathbf{a}_x + 0 \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{H} &= -\frac{20\beta}{\mu} \int \cos(10^8 t - \beta z) dt \mathbf{a}_x \\ &= -\frac{20\beta}{\mu 10^8} \sin(10^8 t - \beta z) \mathbf{a}_x \end{aligned} \quad (9.8.1)$$

It is readily verified that

$$\nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} = 0$$

showing that Gauss's law for magnetic fields is satisfied. Lastly, from Ampere's law

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \quad \mathbf{E} = \frac{1}{\varepsilon} \int (\nabla \times \mathbf{H}) dt \quad (9.8.2)$$

because $\sigma = 0$.

But

$$\begin{aligned} \nabla \times \mathbf{H} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = -\frac{\partial H_x}{\partial z} \mathbf{a}_y - \frac{\partial H_x}{\partial y} \mathbf{a}_z \\ &= \frac{20\beta^2}{\mu 10^8} \cos(10^8 t - \beta z) \mathbf{a}_y + 0 \end{aligned}$$

where $\mathbf{H}$ in eq. (9.8.1) has been substituted. Thus eq. (9.8.2) becomes

$$\begin{aligned} \mathbf{E} &= \frac{20\beta^2}{\mu \varepsilon 10^8} \int \cos(10^8 t - \beta z) dt \mathbf{a}_y \\ &= \frac{20\beta^2}{\mu \varepsilon 10^{16}} \sin(10^8 t - \beta z) \mathbf{a}_y \end{aligned}$$

Comparing this with the given $\mathbf{E}$, we have

$$\frac{20\beta^2}{\mu \varepsilon 10^{16}} = 20$$

or

$$\begin{aligned}\beta &= \pm 10^8 \sqrt{\mu \epsilon} = \pm 10^8 \sqrt{\mu_0 \cdot 4\epsilon_0} = \pm \frac{10^8(2)}{c} = \pm \frac{10^8(2)}{3 \times 10^8} \\ &= \pm \frac{2}{3}\end{aligned}$$

From eq. (9.8.1),

$$\mathbf{H} = \pm \frac{20(2/3)}{4\pi \cdot 10^{-7}(10^8)} \sin\left(10^8 t \pm \frac{2z}{3}\right) \mathbf{a}_x$$

or

$$\mathbf{H} = \pm \frac{1}{3\pi} \sin\left(10^8 t \pm \frac{2z}{3}\right) \mathbf{a}_x \text{ A/m}$$

Method 2 (using phasors):

$$\mathbf{E} = \text{Im}(E_s e^{j\omega t}) \rightarrow \mathbf{E}_s = 20e^{-j\beta z} \mathbf{a}_y \quad (9.8.3)$$

where $\omega = 10^8$.

Again

$$\nabla \cdot \mathbf{E}_s = \frac{\partial E_{ys}}{\partial y} = 0$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \rightarrow \mathbf{H}_s = \frac{\nabla \times \mathbf{E}_s}{-j\omega\mu}$$

or

$$\mathbf{H}_s = \frac{1}{-j\omega\mu} \left[-\frac{\partial E_{ys}}{\partial z} \mathbf{a}_x \right] = -\frac{20\beta}{\omega\mu} e^{-j\beta z} \mathbf{a}_x \quad (9.8.4)$$

Notice that $\nabla \cdot \mathbf{H}_s = 0$ is satisfied.

$$\nabla \times \mathbf{H}_s = j\omega\epsilon\mathbf{E}_s \rightarrow \mathbf{E}_s = \frac{\nabla \times \mathbf{H}_s}{j\omega\epsilon} \quad (9.8.5)$$

Substituting $\mathbf{H}_s$ in eq. (9.8.4) into eq. (9.8.5) gives

$$\mathbf{E}_s = \frac{1}{j\omega\epsilon} \frac{\partial H_{xs}}{\partial z} \mathbf{a}_y = \frac{20\beta^2 e^{-j\beta z}}{\omega^2 \mu \epsilon} \mathbf{a}_y$$

Comparing this with the given $\mathbf{E}_s$ in eq. (9.8.3), we have

$$20 = \frac{20\beta^2}{\omega^2 \mu \epsilon}$$

or

$$\beta = \pm \omega \sqrt{\mu\epsilon} = \pm \frac{2}{3}$$

as obtained before. From eq. (9.8.4),

$$\mathbf{H}_s = \pm \frac{20(2/3) e^{\pm j\beta z}}{10^8(4\pi \times 10^{-7})} \mathbf{a}_x = \pm \frac{1}{3\pi} e^{\pm j\beta z} \mathbf{a}_x$$

$$\begin{aligned} \mathbf{H} &= \text{Im} (\mathbf{H}_s e^{j\omega t}) \\ &= \pm \frac{1}{3\pi} \sin (10^8 t \pm \beta z) \mathbf{a}_x \text{ A/m} \end{aligned}$$

as obtained before. It should be noticed that working with phasors provides a considerable simplification compared with working directly in time domain. Also, notice that we have used

$$\mathbf{A} = \text{Im} (\mathbf{A}_s e^{j\omega t})$$

because the given $\mathbf{E}$ is in sine form and not cosine. We could have used

$$\mathbf{A} = \text{Re} (\mathbf{A}_s e^{j\omega t})$$

in which case sine is expressed in terms of cosine and eq. (9.8.3) would be

$$\mathbf{E} = 20 \cos (10^8 t - \beta z - 90^\circ) \mathbf{a}_y = \text{Re} (\mathbf{E}_s e^{j\omega t})$$

or

$$\mathbf{E}_s = 20 e^{-j\beta z - j90^\circ} \mathbf{a}_y = -j20 e^{-j\beta z} \mathbf{a}_y$$

and we follow the same procedure.

PRACTICE EXERCISE 9.8

A medium is characterized by $\sigma = 0$, $\mu = 2\mu_0$ and $\epsilon = 5\epsilon_0$. If $\mathbf{H} = 2 \cos (\omega t - 3y) \mathbf{a}_z$ A/m, calculate ω and $\mathbf{E}$.

Answer: 2.846×10^8 rad/s, $-476.8 \cos (2.846 \times 10^8 t - 3y) \mathbf{a}_x$ V/m.

SUMMARY

1. In this chapter, we have introduced two fundamental concepts: electromotive force (emf), based on Faraday's experiments, and displacement current, which resulted from Maxwell's hypothesis. These concepts call for modifications in Maxwell's curl equations obtained for static EM fields to accommodate the time dependence of the fields.
2. Faraday's law states that the induced emf is given by ($N = 1$)

$$V_{\text{emf}} = -\frac{\partial \Psi}{\partial t}$$

For transformer emf, $V_{\text{emf}} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$

and for motional emf, $V_{\text{emf}} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$.

3. The displacement current

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{S}$$

where $\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$ (displacement current density), is a modification to Ampere's circuit law. This modification attributed to Maxwell predicted electromagnetic waves several years before it was verified experimentally by Hertz.

4. In differential form, Maxwell's equations for dynamic fields are:

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Each differential equation has its integral counterpart (see Tables 9.1 and 9.2) that can be derived from the differential form using Stokes's or divergence theorem. Any EM field must satisfy the four Maxwell's equations simultaneously.

5. Time-varying electric scalar potential $V(x, y, z, t)$ and magnetic vector potential $\mathbf{A}(x, y, z, t)$ are shown to satisfy wave equations if Lorentz's condition is assumed.
6. Time-harmonic fields are those that vary sinusoidally with time. They are easily expressed in phasors, which are more convenient to work with. Using the cosine reference, the instantaneous vector quantity $\mathbf{A}(x, y, z, t)$ is related to its phasor form $\mathbf{A}_s(x, y, z)$ according to

$$\mathbf{A}(x, y, z, t) = \text{Re} [\mathbf{A}_s(x, y, z) e^{j\omega t}]$$

REVIEW QUESTIONS

- 9.1 The flux through each turn of a 100-turn coil is $(t^3 - 2t)$ mWb, where t is in seconds. The induced emf at $t = 2$ s is

- (a) 1 V
- (b) -1 V
- (c) 4 mV
- (d) 0.4 V
- (e) -0.4 V

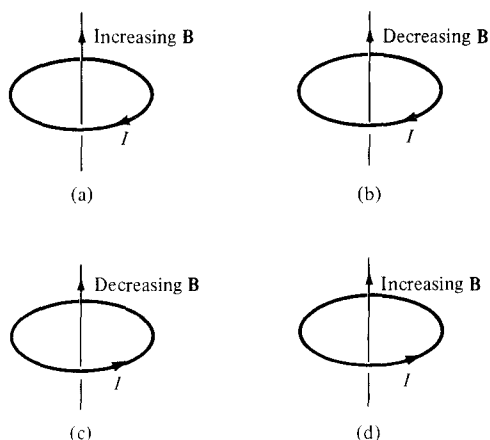


Figure 9.13 For Review Question 9.2.

- 9.2 Assuming that each loop is stationary and the time-varying magnetic field $\mathbf{B}$ induces current I , which of the configurations in Figure 9.13 are incorrect?
- 9.3 Two conducting coils 1 and 2 (identical except that 2 is split) are placed in a uniform magnetic field that decreases at a constant rate as in Figure 9.14. If the plane of the coils is perpendicular to the field lines, which of the following statements is true?
- An emf is induced in both coils.
 - An emf is induced in split coil 2.
 - Equal joule heating occurs in both coils.
 - Joule heating does not occur in either coil.
- 9.4 A loop is rotating about the y -axis in a magnetic field $\mathbf{B} = B_0 \sin \omega t \mathbf{a}_x$ Wb/m². The voltage induced in the loop is due to
- Motional emf
 - Transformer emf
 - A combination of motional and transformer emf
 - None of the above
- 9.5 A rectangular loop is placed in the time-varying magnetic field $\mathbf{B} = 0.2 \cos 150\pi t \mathbf{a}_z$ Wb/m² as shown in Figure 9.15. V_1 is not equal to V_2 .
- True
 - False

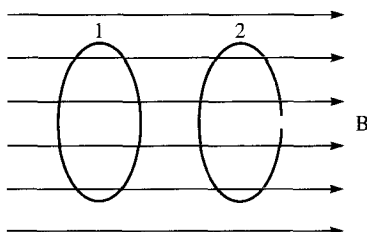
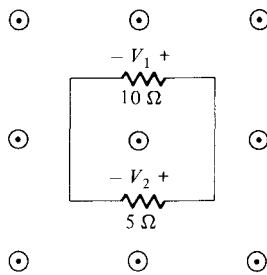


Figure 9.14 For Review Question 9.3.

Figure 9.15 For Review Question 9.5 and Problem 9.10.



9.6 The concept of displacement current was a major contribution attributed to

- (a) Faraday
- (b) Lenz
- (c) Maxwell
- (d) Lorentz
- (e) Your professor

9.7 Identify which of the following expressions are not Maxwell's equations for time-varying fields:

- (a) $\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0$
- (b) $\nabla \cdot \mathbf{D} = \rho_v$
- (c) $\nabla \cdot \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
- (d) $\oint \mathbf{H} \cdot d\mathbf{l} = \int \left(\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{S}$
- (e) $\oint \mathbf{B} \cdot d\mathbf{S} = 0$

9.8 An EM field is said to be nonexistent or not Maxwellian if it fails to satisfy Maxwell's equations and the wave equations derived from them. Which of the following fields in free space are not Maxwellian?

- (a) $\mathbf{H} = \cos x \cos 10^6 t \mathbf{a}_x$
- (b) $\mathbf{E} = 100 \cos \omega t \mathbf{a}_x$
- (c) $\mathbf{D} = e^{-10y} \sin(10^5 - 10y) \mathbf{a}_z$
- (d) $\mathbf{B} = 0.4 \sin 10^4 t \mathbf{a}_z$
- (e) $\mathbf{H} = 10 \cos \left(10^5 t - \frac{z}{10} \right) \mathbf{a}_x$
- (f) $\mathbf{E} = \frac{\sin \theta}{r} \cos(\omega t - r\omega \sqrt{\mu_0 \varepsilon_0}) \mathbf{a}_\theta$
- (g) $\mathbf{B} = (1 - \rho^2) \sin \omega t \mathbf{a}_z$

- 9.9** Which of the following statements is not true of a phasor?
- (a) It may be a scalar or a vector.
 - (b) It is a time-dependent quantity.
 - (c) A phasor V_s may be represented as $V_o \angle \theta$ or $V_o e^{j\theta}$ where $V_o = |V_s|$.
 - (d) It is a complex quantity.
- 9.10** If $\mathbf{E}_s = 10 e^{j4x} \mathbf{a}_y$, which of these is not a correct representation of $\mathbf{E}$?
- (a) $\text{Re}(\mathbf{E}_s e^{j\omega t})$
 - (b) $\text{Re}(\mathbf{E}_s e^{-j\omega t})$
 - (c) $\text{Im}(\mathbf{E}_s e^{j\omega t})$
 - (d) $10 \cos(\omega t + j4x) \mathbf{a}_y$
 - (e) $10 \sin(\omega t + 4x) \mathbf{a}_y$

Answers: 9.1b, 9.2b, d, 9.3a, 9.4c, 9.5a, 9.6c, 9.7a, b, d, g, 9.8b, 9.9a,c, 9.10d.

PROBLEMS

- 9.1** A conducting circular loop of radius 20 cm lies in the $z = 0$ plane in a magnetic field $\mathbf{B} = 10 \cos 377t \mathbf{a}_z$ mWb/m². Calculate the induced voltage in the loop.
- 9.2** A rod of length ℓ rotates about the z -axis with an angular velocity ω . If $\mathbf{B} = B_0 \mathbf{a}_z$, calculate the voltage induced on the conductor.
- 9.3** A 30-cm by 40-cm rectangular loop rotates at 130 rad/s in a magnetic field 0.06 Wb/m² normal to the axis of rotation. If the loop has 50 turns, determine the induced voltage in the loop.
- 9.4** Figure 9.16 shows a conducting loop of area 20 cm² and resistance 4 Ω . If $\mathbf{B} = 40 \cos 10^4 t \mathbf{a}_z$ mWb/m², find the induced current in the loop and indicate its direction.
- 9.5** Find the induced emf in the V-shaped loop of Figure 9.17. (a) Take $\mathbf{B} = 0.1 \mathbf{a}_z$ Wb/m² and $\mathbf{u} = 2 \mathbf{a}_x$ m/s and assume that the sliding rod starts at the origin when $t = 0$. (b) Repeat part (a) if $\mathbf{B} = 0.5x \mathbf{a}_z$ Wb/m².

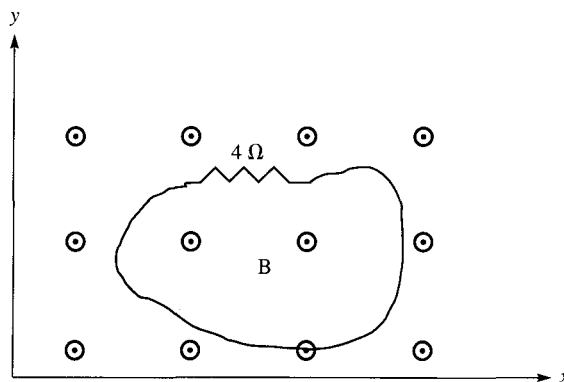
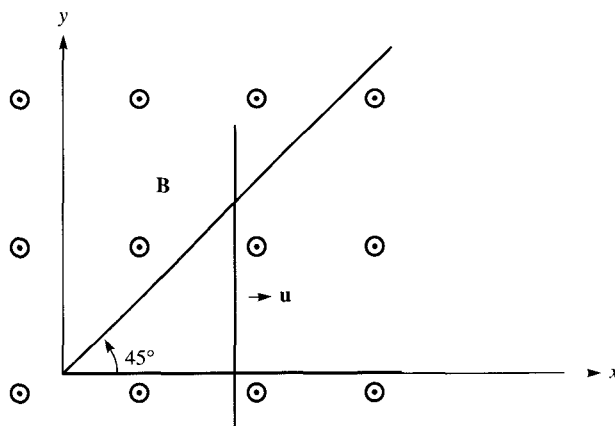


Figure 9.16 For Problem 9.4.

Figure 9.17 For Problem 9.5.

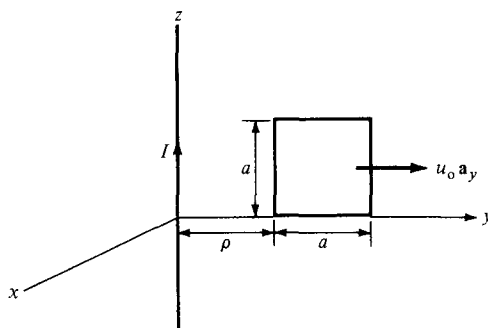


- *9.6** A square loop of side a recedes with a uniform velocity $u_0 \mathbf{a}_y$ from an infinitely long filament carrying current I along $\mathbf{a}_z$ as shown in Figure 9.18. Assuming that $\rho = \rho_0$ at time $t = 0$, show that the emf induced in the loop at $t > 0$ is

$$V_{\text{emf}} = \frac{u_0 a^2 \mu_0 I}{2\pi \rho(\rho + a)}$$

- *9.7** A conducting rod moves with a constant velocity of $3\mathbf{a}_z$ m/s parallel to a long straight wire carrying current 15 A as in Figure 9.19. Calculate the emf induced in the rod and state which end is at higher potential.
- *9.8** A conducting bar is connected via flexible leads to a pair of rails in a magnetic field $\mathbf{B} = 6 \cos 10t \mathbf{a}_x$ mWb/m² as in Figure 9.20. If the z -axis is the equilibrium position of the bar and its velocity is $2 \cos 10t \mathbf{a}_y$ m/s, find the voltage induced in it.
- 9.9** A car travels at 120 km/hr. If the earth's magnetic field is 4.3×10^{-5} Wb/m², find the induced voltage in the car bumper of length 1.6 m. Assume that the angle between the earth magnetic field and the normal to the car is 65° .
- *9.10** If the area of the loop in Figure 9.15 is 10 cm², calculate V_1 and V_2 .

Figure 9.18 For Problem 9.6.



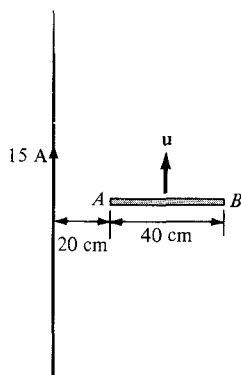


Figure 9.19 For Problem 9.7.

- 9.11** As portrayed in Figure 9.21, a bar magnet is thrust toward the center of a coil of 10 turns and resistance $15\ \Omega$. If the magnetic flux through the coil changes from $0.45\ \text{Wb}$ to $0.64\ \text{Wb}$ in $0.02\ \text{s}$, what is the magnitude and direction (as viewed from the side near the magnet) of the induced current?
- 9.12** The cross section of a homopolar generator disk is shown in Figure 9.22. The disk has inner radius $\rho_1 = 2\ \text{cm}$ and outer radius $\rho_2 = 10\ \text{cm}$ and rotates in a uniform magnetic field $15\ \text{mWb/m}^2$ at a speed of $60\ \text{rad/s}$. Calculate the induced voltage.
- 9.13** A 50-V voltage generator at $20\ \text{MHz}$ is connected to the plates of an air dielectric parallel-plate capacitor with plate area $2.8\ \text{cm}^2$ and separation distance $0.2\ \text{mm}$. Find the maximum value of displacement current density and displacement current.
- 9.14** The ratio J/J_d (conduction current density to displacement current density) is very important at high frequencies. Calculate the ratio at $1\ \text{GHz}$ for:
- distilled water ($\mu = \mu_0, \epsilon = 81\epsilon_0, \sigma = 2 \times 10^{-3}\ \text{S/m}$)
 - sea water ($\mu = \mu_0, \epsilon = 81\epsilon_0, \sigma = 25\ \text{S/m}$)
 - limestone ($\mu = \mu_0, \epsilon = 5\epsilon_0, \sigma = 2 \times 10^{-4}\ \text{S/m}$)
- 9.15** Assuming that sea water has $\mu = \mu_0, \epsilon = 81\epsilon_0, \sigma = 20\ \text{S/m}$, determine the frequency at which the conduction current density is 10 times the displacement current density in magnitude.

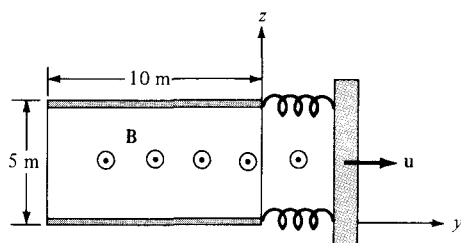


Figure 9.20 For Problem 9.8.

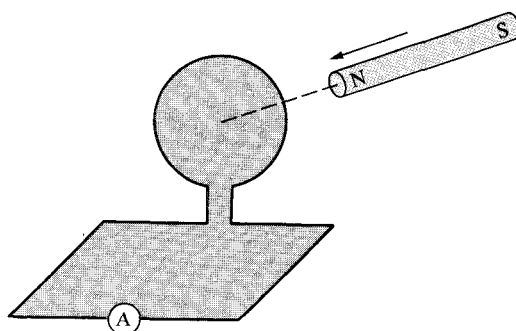


Figure 9.21 For Problem 9.11.

- 9.16 A conductor with cross-sectional area of 10 cm^2 carries a conduction current $0.2 \sin 10^9 t \text{ mA}$. Given that $\sigma = 2.5 \times 10^6 \text{ S/m}$ and $\epsilon_r = 6$, calculate the magnitude of the displacement current density.
- 9.17 (a) Write Maxwell's equations for a linear, homogeneous medium in terms of $\mathbf{E}_s$ and $\mathbf{H}_s$ only assuming the time factor $e^{-j\omega t}$.
 (b) In Cartesian coordinates, write the point form of Maxwell's equations in Table 9.2 as eight scalar equations.
- 9.18 Show that in a source-free region ($\mathbf{J} = 0, \rho_v = 0$), Maxwell's equations can be reduced to two. Identify the two all-embracing equations.
- 9.19 In a linear homogeneous and isotropic conductor, show that the charge density ρ_v satisfies

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon} \rho_v = 0$$

- 9.20 Assuming a source-free region, derive the diffusion equation

$$\nabla^2 \mathbf{E} = \mu \sigma \frac{\partial \mathbf{E}}{\partial t}$$

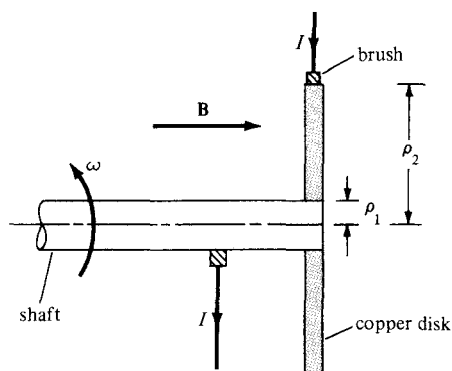


Figure 9.22 For Problem 9.12.

9.21 In a certain region,

$$\mathbf{J} = (2y\mathbf{a}_x + xz\mathbf{a}_y + z^3\mathbf{a}_z) \sin 10^4 t \text{ A/m}$$

find ρ_v if $\rho_v(x, y, 0, t) = 0$.

9.22 In a charge-free region for which $\sigma = 0$, $\epsilon = \epsilon_0\epsilon_r$, and $\mu = \mu_0$,

$$\mathbf{H} = 5 \cos(10^{11}t - 4y)\mathbf{a}_z \text{ A/m}$$

find: (a) $\mathbf{J}_d$ and $\mathbf{D}$, (b) ϵ_r .

9.23 In a certain region with $\sigma = 0$, $\mu = \mu_0$, and $\epsilon = 6.25\epsilon_0$, the magnetic field of an EM wave is

$$\mathbf{H} = 0.6 \cos \beta x \cos 10^8 t \mathbf{a}_z \text{ A/m}$$

Find β and the corresponding $\mathbf{E}$ using Maxwell's equations.

***9.24** In a nonmagnetic medium,

$$\mathbf{E} = 50 \cos(10^9 t - 8x)\mathbf{a}_y + 40 \sin(10^9 t - 8x)\mathbf{a}_z \text{ V/m}$$

find the dielectric constant ϵ_r and the corresponding $\mathbf{H}$.

9.25 Check whether the following fields are genuine EM fields, i.e., they satisfy Maxwell's equations. Assume that the fields exist in charge-free regions.

(a) $\mathbf{A} = 40 \sin(\omega t + 10x)\mathbf{a}_z$

(b) $\mathbf{B} = \frac{10}{\rho} \cos(\omega t - 2\rho)\mathbf{a}_\phi$

(c) $\mathbf{C} = \left(3\rho^2 \cot \phi \mathbf{a}_\rho + \frac{\cos \phi}{\rho} \mathbf{a}_\phi \right) \sin \omega t$

(d) $\mathbf{D} = \frac{1}{r} \sin \theta \sin(\omega t - 5r)\mathbf{a}_\theta$

****9.26** Given the total electromagnetic energy

$$W = \frac{1}{2} \int (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) dv$$

show from Maxwell's equations that

$$\frac{\partial W}{\partial t} = - \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} - \int_V \mathbf{E} \cdot \mathbf{J} dv$$

9.27 In free space,

$$\mathbf{H} = \rho(\sin \phi \mathbf{a}_\rho + 2 \cos \phi \mathbf{a}_\phi) \cos 4 \times 10^6 t \text{ A/m}$$

find $\mathbf{J}_d$ and $\mathbf{E}$.

9.28 An antenna radiates in free space and

$$\mathbf{H} = \frac{12 \sin \theta}{r} \cos(2\pi \times 10^8 t - \beta r) \mathbf{a}_\theta \text{ mA/m}$$

find the corresponding $\mathbf{E}$ in terms of β .

***9.29** The electric field in air is given by $\mathbf{E} = \rho t e^{-\rho - t} \mathbf{a}_\phi$ V/m; find $\mathbf{B}$ and $\mathbf{J}$.

****9.30** In free space ($\rho_v = 0$, $\mathbf{J} = 0$). Show that

$$\mathbf{A} = \frac{\mu_0}{4\pi r} (\cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta) e^{j\omega(t-r/c)}$$

satisfies the wave equation in eq. (9.52). Find the corresponding V . Take c as the speed of light in free space.

9.31 Evaluate the following complex numbers and express your answers in polar form:

(a) $(4 \angle 30^\circ - 10 \angle 50^\circ)^{1/2}$

(b) $\frac{1 + j2}{6 + j8 - 7 \angle 15^\circ}$

(c) $\frac{(3 + j4)^2}{12 - j7 + (-6 + j10)^*}$

(d) $\frac{(3.6 \angle -200^\circ)^{1/2}}{(2.4 \angle 45^\circ)^2 (-5 + j8)^*}$

9.32 Write the following time-harmonic fields as phasors:

(a) $\mathbf{E} = 4 \cos(\omega t - 3x - 10^\circ) \mathbf{a}_y - \sin(\omega t + 3x + 20^\circ) \mathbf{a}_z$

(b) $\mathbf{H} = \frac{\sin \theta}{r} \cos(\omega t - 5r) \mathbf{a}_\theta$

(c) $\mathbf{J} = 6e^{-3x} \sin(\omega t - 2x) \mathbf{a}_y + 10e^{-x} \cos(\omega t - 5x) \mathbf{a}_z$

9.33 Express the following phasors in their instantaneous forms:

(a) $\mathbf{A}_s = (4 - 3j)e^{-j\beta x} \mathbf{a}_y$

(b) $\mathbf{B}_s = \frac{20}{\rho} e^{-j2z} \mathbf{a}_\rho$

(c) $\mathbf{C}_s = \frac{10}{r^2} (1 + j2)e^{-j\phi} \sin \theta \mathbf{a}_\phi$

9.34 Given $\mathbf{A} = 4 \sin \omega t \mathbf{a}_x + 3 \cos \omega t \mathbf{a}_y$ and $\mathbf{B}_s = j10ze^{-jz} \mathbf{a}_x$, express $\mathbf{A}$ in phase form and $\mathbf{B}_s$ in instantaneous form.

9.35 Show that in a linear homogeneous, isotropic source-free region, both $\mathbf{E}_s$ and $\mathbf{H}_s$ must satisfy the wave equation

$$\nabla^2 \mathbf{A}_s + \gamma^2 \mathbf{A}_s = 0$$

where $\gamma^2 = \omega^2 \mu \epsilon - j\omega \mu \sigma$ and $\mathbf{A}_s = \mathbf{E}_s$ or $\mathbf{H}_s$.