Chapter 15

NUMERICAL METHODS

The recipe for ignorance is: be satisfied with your opinions and content with your knowledge.

—ELBERT HUBBARD

15.1 INTRODUCTION

In the preceding chapters we considered various analytic techniques for solving EM problems and obtaining solutions in closed form. A *closed form solution* is one in the form of an explicit, algebraic equation in which values of the problem parameters can be substituted. Some of these analytic solutions were obtained assuming certain situations, thereby making the solutions applicable to those idealized situations. For example, in deriving the formula for calculating the capacitance of a parallel-plate capacitor, we assumed that the fringing effect was negligible and that the separation distance was very small compared with the width and length of the plates. Also, our application of Laplace's equation in Chapter 6 was restricted to problems with boundaries coinciding with coordinate surfaces. Analytic solutions have an inherent advantage of being exact. They also make it easy to observe the behavior of the solution for variation in the problem parameters. However, analytic solutions are available only for problems with simple configurations.

When the complexities of theoretical formulas make analytic solution intractable, we resort to nonanalytic methods, which include (1) graphical methods, (2) experimental methods, (3) analog methods, and (4) numerical methods. Graphical, experimental, and analog methods are applicable to solving relatively few problems. Numerical methods have come into prominence and become more attractive with the advent of fast digital computers. The three most commonly used simple numerical techniques in EM are (1) moment method, (2) finite difference method, and (3) finite element method. Most EM problems involve either partial differential equations or integral equations. Partial differential equations are usually solved using the finite difference method or the finite element method; integral equations are solved conveniently using the moment method. Although numerical methods give approximate solutions, the solutions are sufficiently accurate for engineering purposes. We should not get the impression that analytic techniques are outdated because of numerical methods; rather they are complementary. As will be observed later, every numerical method involves an analytic simplification to the point where it is easy to apply the method.

The Matlab codes developed for computer implementation of the concepts developed in this chapter are simplified and self-explanatory for instructional purposes. The notations

used in the programs are as close as possible to those used in the main text; some are defined wherever necessary. These programs are by no means unique; there are several ways of writing a computer program. Therefore, users may decide to modify the programs to suit their objectives.

†15.2 FIELD PLOTTING

In Section 4.9, we used field lines and equipotential surfaces for visualizing an electrostatic field. However, the graphical representations in Figure 4.21 for electrostatic fields and in Figures 7.8(b) and 7.16 for magnetostatic fields are very simple, trivial, and qualitative. Accurate pictures of more complicated charge distributions would be more helpful. In this section, a numerical technique that may be developed into an interactive computer program is presented. It generates data points for electric field lines and equipotential lines for arbitrary configuration of point sources.

Electric field lines and equipotential lines can be plotted for coplanar point sources with simple programs. Suppose we have N point charges located at position vectors $\mathbf{r}_1$, $\mathbf{r}_2, \ldots, \mathbf{r}_N$, the electric field intensity $\mathbf{E}$ and potential V at position vector $\mathbf{r}$ are given, respectively, by

$$\mathbf{E} = \sum_{k=1}^{N} \frac{Q_k (\mathbf{r} - \mathbf{r}_k)}{4\pi\varepsilon |\mathbf{r} - \mathbf{r}_k|^3} \tag{15.1}$$

and

$$V = \sum_{k=1}^{N} \frac{Q_k}{4\pi\varepsilon |\mathbf{r} - \mathbf{r}_k|} \tag{15.2}$$

If the charges are on the same plane (z = constant), eqs. (15.1) and (15.2) become

$$\mathbf{E} = \sum_{k=1}^{N} \frac{Q_k[(x - x_k)\mathbf{a}_x + (y - y_k)\mathbf{a}_y]}{4\pi\varepsilon[(x - x_k)^2 + (y - y_k)^2]^{3/2}} \tag{15.3}$$

$$V = \sum_{k=1}^{N} \frac{Q_k}{4\pi\varepsilon[(x - x_k)^2 + (y - y_k)^2]^{1/2}} \tag{15.4}$$

To plot the electric field lines, follow these steps:

1. Choose a starting point on the field line.
2. Calculate E_x and E_y at that point using eq. (15.3).
3. Take a small step along the field line to a new point in the plane. As shown in Figure 15.1, a movement $\Delta\ell$ along the field line corresponds to movements Δx and Δy along x- and y-directions, respectively. From the figure, it is evident that

$$\frac{\Delta x}{\Delta \ell} = \frac{E_x}{E} = \frac{E_x}{[E_x^2 + E_y^2]^{1/2}}$$

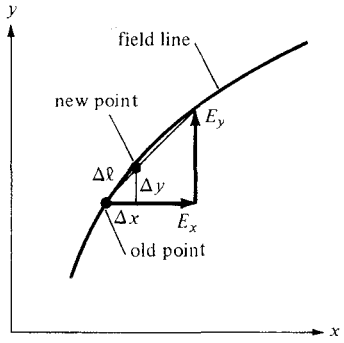


Figure 15.1 A small displacement on a field line.

or

$$\Delta x = \frac{\Delta \ell \cdot E_x}{[E_x^2 + E_y^2]^{1/2}} \tag{15.5}$$

Similarly,

$$\Delta y = \frac{\Delta \ell \cdot E_y}{[E_x^2 + E_y^2]^{1/2}} \tag{15.6}$$

Move along the field line from the old point (x, y) to a new point $x' = x + \Delta x$, $y' = y + \Delta y$.

4. Go back to steps 2 and 3 and repeat the calculations. Continue to generate new points until a line is completed within a given range of coordinates. On completing the line, go back to step 1 and choose another starting point. Note that since there are an infinite number of field lines, any starting point is likely to be on a field line. The points generated can be plotted by hand or by a plotter as illustrated in Figure 15.2.

To plot the equipotential lines, follow these steps:

1. Choose a starting point.
2. Calculate the electric field (E_x, E_y) at that point using eq. (15.3).

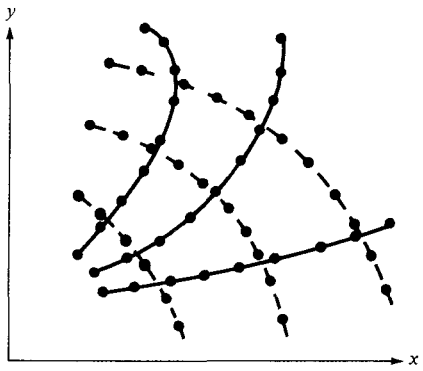


Figure 15.2 Generated points on E-field lines (shown thick) and equipotential lines (shown dotted).

3. Move a small step along the line perpendicular to E-field line at that point. Utilize the fact that if a line has slope m, a perpendicular line must have slope $-1/m$. Since an E-field line and an equipotential line meeting at a given point are mutually orthogonal there,

$$\Delta x = \frac{-\Delta\ell \cdot E_y}{[E_x^2 + E_y^2]^{1/2}} \tag{15.7}$$

$$\Delta y = \frac{\Delta\ell \cdot E_x}{[E_x^2 + E_y^2]^{1/2}} \tag{15.8}$$

Move along the equipotential line from the old point (x, y) to a new point $(x + \Delta x, y + \Delta y)$. As a way of checking the new point, calculate the potential at the new and old points using eq. (15.4); they must be equal because the points are on the same equipotential line.

4. Go back to steps 2 and 3 and repeat the calculations. Continue to generate new points until a line is completed within the given range of x and y. After completing the line, go back to step 1 and choose another starting point. Join the points generated by hand or by a plotter as illustrated in Figure 15.2.

By following the same reasoning, the magnetic field line due to various current distributions can be plotted using Biot–Savart law. Programs for determining the magnetic field line due to line current, a current loop, a Helmholtz pair, and a solenoid can be developed. Programs for drawing the electric and magnetic field lines inside a rectangular waveguide or the power radiation pattern produced by a linear array of vertical half-wave electric dipole antennas can also be written.

EXAMPLE 15.1

Write a program to plot the electric field and equipotential lines due to:

(a) Two point charges Q and $-4Q$ located at $(x, y) = (-1, 0)$ and $(1, 0)$, respectively.

(b) Four point charges Q, $-Q$, Q, and $-Q$ located at $(x, y) = (-1, -1), (1, -1), (1, 1)$, and $(-1, 1)$, respectively. Take $Q/4\pi\varepsilon = 1$ and $\Delta\ell = 0.1$. Consider the range $-5 \leq x, y \leq 5$.

Solution:

Based on the steps given in Section 15.2, the program in Figure 15.3 was developed. Enough comments are inserted to make the program as self-explanatory as possible. For example, to use the program to generate the plot in Figure 15.4(a), load program **plotit** in your Matlab directory. At the command prompt in Matlab, type

plotit ([1 −4], [−1 0; 1 0], 1, 1, 0.1, 0.01, 8, 2, 5)

where the numbers have meanings provided in the program. Further explanation of the program is provided in the following paragraphs.

Since the E-field lines emanate from positive charges and terminate on negative charges, it seems reasonable to generate starting points (x_s, y_s) for the E-field lines on small circles centered at charge locations (x_Q, y_Q); that is,

$$x_s = x_Q + r\cos\theta \tag{15.1.1a}$$

$$y_s = y_Q + r\sin\theta \tag{15.1.1b}$$

```
function plotit(charges,location,ckEField,ckEq,DLE,DLV,NLE,NLV,PTS)
figure;
hold on;
% Program for plotting the electric field lines
% and equipotential lines due to coplanar point charges
% the plot is to be within the range -5<x,y<5
%
% This is the correct usage:
% function plotit(charges,location,ckEField,ckEq,DLE,DLV,NLE,NLV,PTS)
%
% where,
%    charges = a vector containing the charges
%   location = a matrix where each row is a charge location
%   ckEField = Flag set to 1 plots the Efield lines
%       ckEq = Flag set to 1 plots the Equipotential lines
% DLE or DLV = the increment along E & V lines
%        NLE = No. of E-Field lines per charge
%        NLV = No. of Equipotential lines per charge
%        PTS => Plots every PTS point (i.e. if PTS = 5 then plot
every 5th point)
% note that constant Q/4*Pie*ErR is set equal to 1.0

% Determine the E-Field LInes
% For convenience, the starting points (XS,YS) are radially
distributed about charge locations
Q=charges;
XQ = location(:,1);
YQ = location(:,2);
JJ=1;
NQ = length(charges);
if (ckEField)
for K=1:NQ
   for I =1:NLE
      THETA = 2*pi*(I-1)/(NLE);
      XS=XQ(K) + 0.1*cos(THETA);
      YS=YQ(K) + 0.1*sin(THETA);
      XE=XS;
      YE=YS;
      JJ=JJ+1;
      if (~mod(JJ,PTS))
         plot (XE,YE);
      end
      while(1)
      % Find i ncrement and new point (X,Y)
         EX=0;
         EY=0;
```

Figure 15.3 Computer program for Example 15.1.

```
            for J=1:NQ
               R =sqrt((XE-XQ(J))^2 + (YE - YQ(J)) ^2);
               EX = EX +Q(J)*(XE-XQ(J))/(R^3);
               EY = EY +Q(J)*(YE-YQ(J))/(R^3);
            end
            E = sqrt(EX^2 + EY^2);

            % CHECK FOR A SINGULAR POINT
            if (E <=.00005)
               break;
            end
            DX = DLE*EX/E;
            DY = DLE*EY/E;
            % FOR NEGATIVE CHARGE, NEGATE DX & DY SO THAT INCREMENT
            % IS AWAY FROM THE CHARGE
            if (Q(K) < 0)
               DX = -DX;
               DY = -DY;
            end
            XE = XE + DX;
            YE = YE + DY;
            % CHECK WHETHER NEW POINT IS WITHIN THE GIVEN RANGE OR
            TOO
            % CLOSE TO ANY OF THE POINT CHARGES - TO AVOID SINGULAR
            POINT
            if ((abs(XE) >= 5) | (abs(YE) >= 5))
               break;
            end

            if (sum(abs(XE-XQ) < .05 & abs(YE-YQ) < .05) >0)
               break;
            end
            JJ=JJ+1;
            if (~mod(JJ,PTS))
               plot (XE,YE);
            end
         end % while loop
      end % I =1:NLE
   end   % K = 1:NQ
end % if
% NEXT, DETERMINE THE EQUIPOTENTIAL LINES
% FOR CONVENIENCE, THE STARTING POINTS (XS,YS) ARE
% CHOSEN LIKE THOSE FOR THE E-FIELD LINES
if (ckEq)
JJ=1;
DELTA = .2;
ANGLE = 45*pi/180;
```

Figure 15.3 (Continued)

```
for K =1:NQ
   FACTOR = .5;
   for KK = 1:NLV
     XS = XQ(K) + FACTOR*cos(ANGLE);
     YS = YQ(K) + FACTOR*sin(ANGLE);
     if ( abs(XS) >= 5 | abs(YS) >=5)
        break;
     end
     DIR = 1;
     XV = XS;
     YV = YS;
     JJ=JJ+1;
     if (~mod(JJ,PTS))
        plot(XV,YV);
     end
% FIND INCREMENT AND NEW POINT (XV,YV)
     N=1;
     while (1)
         EX = 0;
         EY = 0;
         for J = 1:NQ
            R = sqrt((XV-XQ(J))^2 + (YV-YQ(J))^2);
            EX = EX + Q(J)*(XV-XQ(J))/(R^3);
            EY = EY + Q(J)*(YV-YQ(J))/(R^3);
         end
         E=sqrt(EX^2 + EY^2);
         if (E <= .00005)
            FACTOR = 2*FACTOR;
            break;
         end
         DX = -DLV*EY/E;
         DY = DLV*EV/E;
         XV = XV + DIR*DX;
         YV = YV + DIR*DY;
         % CHECK IF THE EQUIPOTENTIAL LINE LOOPS BACK TO (X,YS)
         R0 = sqrt((XV - XS)^2 + (YV - YS)^2);
         if (R0 < DELTA & N < 50)
            FACTOR = 2*FACTOR;
            break;
         end
         % CHECK WHETHER NEW POINT IS WITHIN THE GIVEN RANGE
         % IF FOUND OUT OF RANGE, GO BACK TO THE STARTING POINT
         % (S,YS)BUT INCREMENT IN THE OPPOSITE DIRECTION
         if (abs(XV) > 5  |  abs(YV) > 5)
            DIR = DIR -2;
            XV = XS;
            YV = YS;
```

Figure 15.3 (Continued)

```
            if (abs(DIR) > 1)
               FACTOR = 2*FACTOR;
               break;
            end
         else
            if (sum(abs(XV-XQ) < .005 & abs(YV-YQ) < .005) >0)
                  break;
            end
          end
          JJ=JJ+1;
          if (~mod(JJ,PTS))
             N=N+1;
             plot(XV,YV);
          end
       end % WHILE loop
    end  % KK
end   % K

end % if
```

Figure 15.3 (Continued)

where r is the radius of the small circle (e.g., $r = 0.1$ or 0.05), and θ is a prescribed angle chosen for each E-field line. The starting points for the equipotential lines can be generated in different ways: along the x- and y-axes, along line $y = x$, and so on. However, to make the program as general as possible, the starting points should depend on the charge locations like those for the E-field lines. They could be chosen using eq. (15.1.1) but with fixed θ (e.g., 45°) and variable r (e.g., 0.5, 1.0, 2.0,. . .).

The value of incremental length $\Delta\ell$ is crucial for accurate plots. Although the smaller the value of $\Delta\ell$, the more accurate the plots, we must keep in mind that the smaller the value of $\Delta\ell$, the more points we generate and memory storage may be a problem. For example, a line may consist of more than 1000 generated points. In view of the large number of points to be plotted, the points are usually stored in a data file and a graphics routine is used to plot the data.

For both the E-field and equipotential lines, different checks are inserted in the program in Figure 15.3:

(a) Check for singular point ($\mathbf{E} = 0$?)

(b) Check whether the point generated is too close to a charge location.

(c) Check whether the point is within the given range of $-5 < x, y < 5$.

(d) Check whether the (equipotential) line loops back to the starting point.

The plot of the points generated for the cases of two point charges and four point charges are shown in Figure 15.4(a) and (b), respectively.

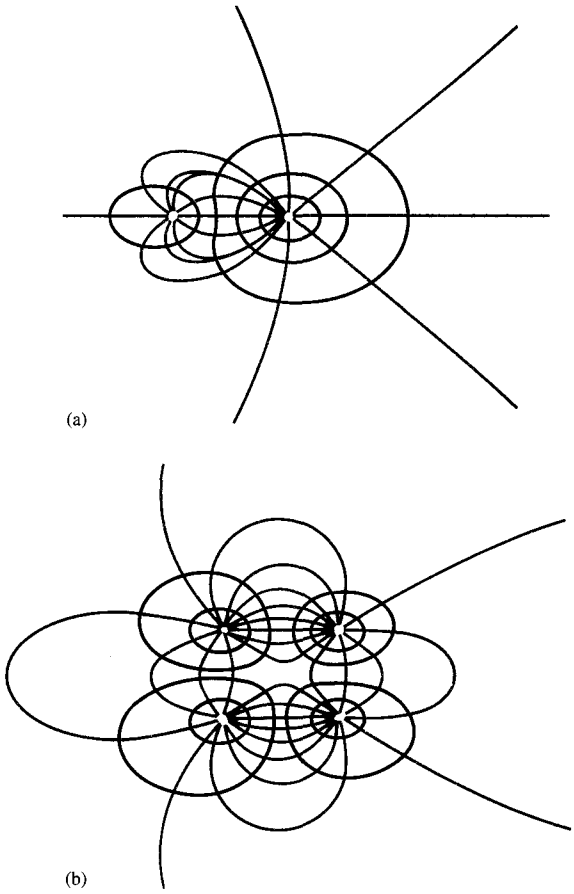


Figure 15.4 For Example 15.1; plots of E-field lines and equipotential lines due to **(a)** two point charges, and **(b)** four point charges (a two-dimensional quadrupole).

PRACTICE EXERCISE 15.1

Write a complete program for plotting the electric field lines and equipotential lines due to coplanar point charges. Run the program for $N = 3$; that is, there are three point charges $-Q$, $+Q$, and $-Q$ located at $(x, y) = (-1, 0)$, $(0, 1)$, and $(1, 0)$ respectively. Take $Q/4\pi\varepsilon = 1$, $\Delta\ell = 0.1$ or 0.01 for greater accuracy and limit your plot to $-5 \leq x, y \leq 5$.

Answer: See Figure 15.5.

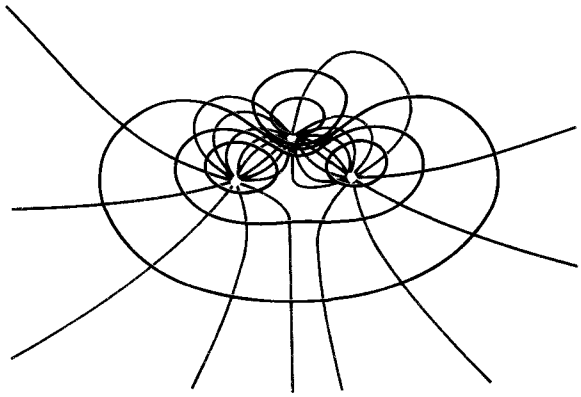


Figure 15.5 For Practice Exercise 15.1.

15.3 THE FINITE DIFFERENCE METHOD

The finite difference method[1] (FDM) is a simple numerical technique used in solving problems like those solved analytically in Chapter 6. A problem is uniquely defined by three things:

1. A partial differential equation such as Laplace's or Poisson's equations
2. A solution region
3. Boundary and/or initial conditions

A finite difference solution to Poisson's or Laplace's equation, for example, proceeds in three steps: (1) dividing the solution region into a grid of nodes, (2) approximating the differential equation and boundary conditions by a set of linear algebraic equations (called *difference equations*) on grid points within the solution region, and (3) solving this set of algebraic equations.

[1]For an extensive treatment of the finite difference method, see G. D. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Methods,* 2nd edition. Oxford: Clarendon, 1978.

Step 1: Suppose we intend to apply the finite difference method to determine the electric potential in a region shown in Figure 15.6(a). The solution region is divided into rectangular meshes with *grid points* or *nodes* as in Figure 15.6(a). A node on the boundary of the region where the potential is specified is called a *fixed node* (fixed by the problem) and interior points in the region are called *free points* (free in that the potential is unknown).

Step 2: Our objective is to obtain the finite difference approximation to Poisson's equation and use this to determine the potentials at all the free points. We recall that Poisson's equation is given by

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \tag{15.9a}$$

For two-dimensional solution region such as in Figure 15.6(a), ρ_v is replaced by ρ_s, $\frac{\partial^2 V}{\partial z^2} = 0$, so

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho_S}{\varepsilon} \tag{15.9b}$$

From the definition of the derivative of $V(x, y)$ at point (x_o, y_o),

$$V' = \frac{\partial V}{\partial x}\bigg|_{x=x_o} \simeq \frac{V(x_o + \Delta x, y_o) - V(x_o - \Delta x, y_o)}{2\Delta x} = \frac{V_{i+1,j} - V_{i-1,j}}{2\,\Delta x} \tag{15.10}$$

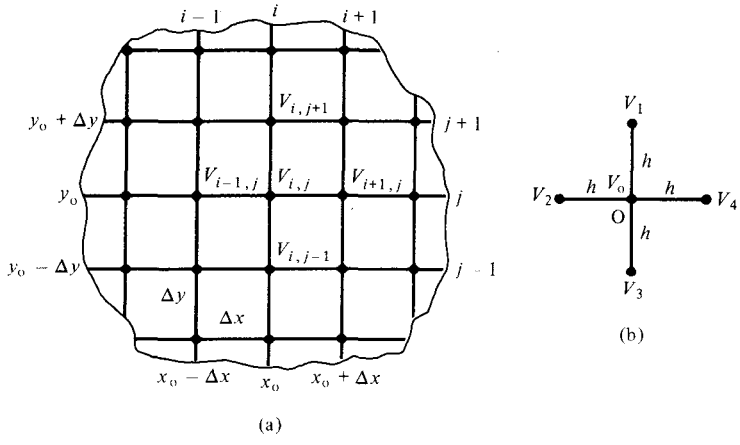


Figure 15.6 Finite difference solution pattern: **(a)** division of the solution into grid points, **(b)** finite difference five-node molecule.

where Δx is a sufficiently small increment along x. For the second derivative, which is the derivative of the first derivative V',

$$V'' = \frac{\partial^2 V}{\partial x^2}\bigg|_{x=x_o} = \frac{\partial V'}{\partial x} \simeq \frac{V'(x_o + \Delta x/2, y_o) - V'(x_o - \Delta x/2, y_o)}{\Delta x}$$
$$= \frac{V(x_o + \Delta x, y_o) - 2V(x_o, y_o) + V(x_o - \Delta x, y_o)}{(\Delta x)^2}$$
$$= \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta x)^2} \tag{15.11}$$

Equations (15.10) and (15.11) are the finite difference approximations for the first and second partial derivatives of V with respect to x, evaluated at $x = x_o$. The approximation in eq. (15.10) is associated with an error of the order of the Δx while that of eq. (15.11) has an associated error on the order of $(\Delta x)^2$. Similarly,

$$\frac{\partial^2 V}{\partial y^2}\bigg|_{y=y_o} \simeq \frac{V(x_o, y_o + \Delta y) - 2V(x_o, y_o) + V(x_o, y_o - \Delta y)}{(\Delta y)^2}$$
$$= \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta y)^2} \tag{15.12}$$

Substituting eqs. (15.11) and (15.12) into eq. (15.9b) and letting $\Delta x = \Delta y = h$ gives

$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = -\frac{h^2 \rho_S}{\varepsilon}$$

or

$$\boxed{V_{i,j} = \frac{1}{4}\left(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} + \frac{h^2 \rho_S}{\varepsilon}\right)} \tag{15.13}$$

where h is called the *mesh size*. Equation (15.13) is the finite difference approximation to Poisson's equation. If the solution region is charge-free ($\rho_S = 0$), eq. (15.9) becomes Laplace's equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{15.14}$$

The finite difference approximation to this equation is obtained from eq. (15.13) by setting $\rho_S = 0$; that is

$$V_{i,j} = \frac{1}{4}(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}) \tag{15.15}$$

This equation is essentially a five-node finite difference approximation for the potential at the central point of a square mesh. Figure 15.6(b) illustrates what is called the finite differ-

ence *five-node molecule.* The molecule in Figure 15.6(b) is taken out of Figure 15.6(a). Thus eq. (15.15) applied to the molecule becomes

$$\boxed{V_o = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)} \tag{15.16}$$

This equation clearly shows the average-value property of Laplace's equation. In other words, Laplace's equation can be interpreted as a differential means of stating the fact that the potential at a specific point is the average of the potentials at the neighboring points.

Step 3: To apply eq. (15.16) [or eq. (15.13)] to a given problem, one of the following two methods is commonly used:

A. Iteration Method

We start by setting initial values of the potentials at the free nodes equal to zero or to any reasonable guessed value. Keeping the potentials at the fixed nodes unchanged at all times, we apply eq. (15.16) to every free node in turn until the potentials at all free nodes are calculated. The potentials obtained at the end of this first iteration are not accurate but just approximate. To increase the accuracy of the potentials, we repeat the calculation at every free node using old values to determine new ones. The iterative or repeated modification of the potential at each free node is continued until a prescribed degree of accuracy is achieved or until the old and the new values at each node are satisfactorily close.

B. Band Matrix Method

Equation (15.16) applied to all free nodes results in a set of simultaneous equations of the form

$$[A]\,[V] = [B] \tag{15.17}$$

where $[A]$ is a *sparse* matrix (i.e., one having many zero terms), $[V]$ consists of the unknown potentials at the free nodes, and $[B]$ is another column matrix formed by the known potentials at the fixed nodes. Matrix $[A]$ is also *banded* in that its nonzero terms appear clustered near the main diagonal because only nearest neighboring nodes affect the potential at each node. The sparse, band matrix is easily inverted to determine $[V]$. Thus we obtain the potentials at the free nodes from matrix $[V]$ as

$$[V] = [A]^{-1}\,[B] \tag{15.18}$$

The finite difference method can be applied to solve time-varying problems. For example, consider the one-dimensional wave equation of eq. (10.1), namely

$$u^2 \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial t^2} \tag{15.19}$$

where u is the wave velocity and Φ is the E- or H-field component of the EM wave. The difference approximations of the derivatives at (x_o, t_o) or (i, j)th node shown in Figure 15.7 are

$$\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{x=x_o} \simeq \frac{\Phi_{i-1,j} - 2\Phi_{i,j} + \Phi_{i+1,j}}{(\Delta x)^2} \tag{15.20}$$

$$\left.\frac{\partial^2 \Phi}{\partial t^2}\right|_{t=t_o} \simeq \frac{\Phi_{i,j-1} - 2\Phi_{i,j} + \Phi_{i,j+1}}{(\Delta t)^2} \tag{15.21}$$

Inserting eqs. (15.20) and (15.21) in eq. (15.20) and solving for $\Phi_{i,j+1}$ gives

$$\boxed{\Phi_{i,j+1} \simeq \alpha(\Phi_{i-1,j} + \Phi_{i+1,j}) + 2(1 - \alpha)\Phi_{i,j} - \Phi_{i,j-1}} \tag{15.22}$$

where

$$\alpha = \left[\frac{u\,\Delta t}{\Delta x}\right]^2 \tag{15.23}$$

It can be shown that for the solution in eq. (15.22) to be stable, $\alpha \leq 1$. To start the finite difference algorithm in eq. (15.22), we use the initial conditions. We assume that at $t = 0$, $\partial \Phi_{i,0}/\partial t = 0$ and use (central) difference approximation (see Review Question 15.2) to get

$$\frac{\partial \Phi_{i,0}}{\partial t} \simeq \frac{\Phi_{i,1} - \Phi_{i,-1}}{2\Delta t} = 0$$

or

$$\Phi_{i,1} = \Phi_{i,-1} \tag{15.24}$$

Substituting eq. (15.24) into eq. (15.22) and taking $j = 0$ ($t = 0$), we obtain

$$\Phi_{i,1} \simeq \alpha(\Phi_{i-1,0} + \Phi_{i+1,0}) + 2(1 - \alpha)\Phi_{i,0} - \Phi_{i,1}$$

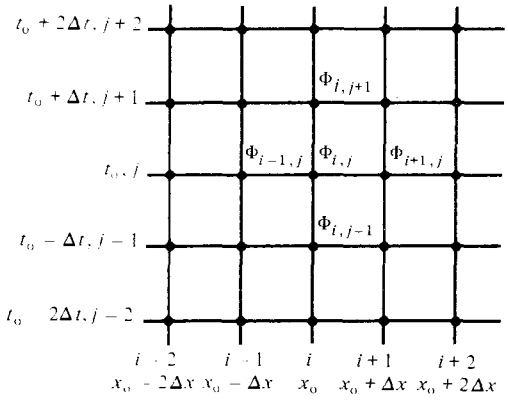


Figure 15.7 Finite difference solution pattern for wave equation.

or

$$\Phi_{i,1} \simeq \frac{1}{2}[\alpha(\Phi_{i-1,0} + \Phi_{i+1,0}) + 2(1 - \alpha)\Phi_{i,0}] \tag{15.25}$$

With eq. (15.25) as the "starting" formula, the value of Φ at any point on the grid can be obtained directly from eq. (15.22). Note that the three methods discussed for solving eq. (15.16) do not apply to eq. (15.22) because eq. (15.22) can be used directly with eq. (15.25) as the starting formula. In other words, we do not have a set of simultaneous equations; eq. (15.22) is an explicit formula.

The concept of FDM can be extended to Poisson's, Laplace's, or wave equations in other coordinate systems. The accuracy of the method depends on the fineness of the grid and the amount of time spent in refining the potentials. We can reduce computer time and increase the accuracy and convergence rate by the method of successive overrelaxation, by making reasonable guesses at initial values, by taking advantage of symmetry if possible, by making the mesh size as small as possible, and by using more complex finite difference molecules (see Figure 15.41). One limitation of the finite difference method is that interpolation of some kind must be used to determine solutions at points not on the grid. One obvious way to overcome this is to use a finer grid, but this would require a greater number of computations and a larger amount of computer storage.

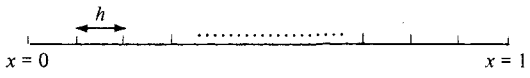
EXAMPLE 15.2

Solve the one-dimensional boundary-value problem $-\Phi'' = x^2$, $0 \le x \le 1$ subject to $\Phi(0) = 0 = \Phi(1)$. Use the finite difference method.

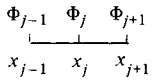
Solution:

First, we obtain the finite difference approximation to the differential equation $\Phi'' = -x^2$, which is Poisson's equation in one dimension. Next, we divide the entire domain $0 \le x \le 1$ into N equal segments each of length $h\ (= 1/N)$ as in Figure 15.8(a) so that there are $(N + 1)$ nodes.

$$-x_o^2 = \left.\frac{d^2\Phi}{dx^2}\right|_{x=x_o} \simeq \frac{\Phi(x_o + h) - 2\Phi(x_o) + \Phi(x_o - h)}{h^2}$$



(a)



(b)

Figure 15.8 For Example 15.2.

or

$$-x_j^2 = \frac{\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}}{h^2}$$

Thus

$$-2\Phi_j = -x_j^2 h^2 - \Phi_{j+1} - \Phi_{j-1}$$

or

$$\Phi_j = \frac{1}{2}(\Phi_{j+1} + \Phi_{j-1} + x_j^2 h^2)$$

Using this finite difference scheme, we obtain an approximate solution for various values of N. The Matlab code is shown in Figure 15.9. The number of iterations NI depends on the degree of accuracy desired. For a one-dimensional problem such as this, $NI = 50$ may suffice. For two- or three-dimensional problems, larger values of NI would be required (see Table 15.1). It should be noted that the values of Φ at end points (fixed nodes) are held fixed. The solutions for $N = 4$ and 10 are shown in Figure 15.10.

We may compare this with the exact solution obtained as follows. Given that $d^2\Phi/dx^2 = -x^2$, integrating twice gives

$$\Phi = -\frac{x^4}{12} + Ax + B$$

```
% ONE-DIMENSIONAL PROBLEM OF EXAMPLE 15.2
% SOLVED USING FINITE DIFFERENCE METHOD
%
% h = MESH SIZE
% ni = NO. OF ITERATIONS DESIRED

      P = [ ];
      n=20;
      ni=1000;
      1=1.0;
      h = 1/n;
      phi=zeros(n+1,1);
      x=h*[0:n]';
      x1=x(2:n);
      for k=1:ni
         phi([2:n])=[phi(3:n+1)+phi(1:n-1)+x1.^2*h^2]/2;
      end
      %  CALCULATE THE EXACT VALUE ALSO
      phiex=x.*(1.0-x.^3)/12.0;
      diary a:test.out
      [[1:n+1]' phi phiex]
      diary off
```

Figure 15.9 Computer program for Example 15.2.

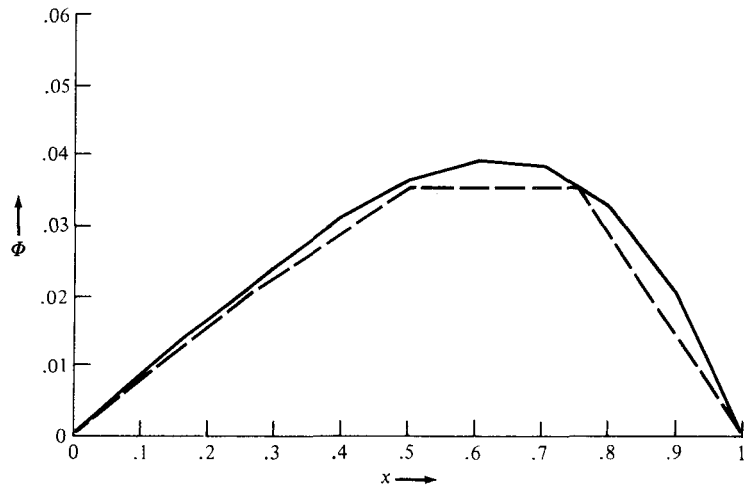


Figure 15.10 For Example 15.2: plot of $\Phi(x)$. Continuous curve is for $N = 10$; dashed curve is for $N = 4$.

where A and B are integration constants. From the boundary conditions,

$$\Phi(0) = 0 \rightarrow B = 0$$

$$\Phi(1) = 0 \rightarrow 0 = -\frac{1}{12} + A \qquad \text{or} \qquad A = \frac{1}{12}$$

Hence, the exact solution is $\Phi = x(1 - x^3)/12$, which is calculated in Figure 15.9 and found to be very close to case $N = 10$.

PRACTICE EXERCISE 15.2

Solve the differential equation $d^2y/dx^2 + y = 0$ with the boundary conditions $y(0) = 0$, $y(1) = 1$ using the finite difference method. Take $\Delta x = 1/4$.

Answer: Compare your result with the exact solution $y(x) = \dfrac{\sin(x)}{\sin(1)}$.

EXAMPLE 15.3

Determine the potential at the free nodes in the potential system of Figure 15.11 using the finite difference method.

Solution:

This problem will be solved using the iteration method and band matrix method.

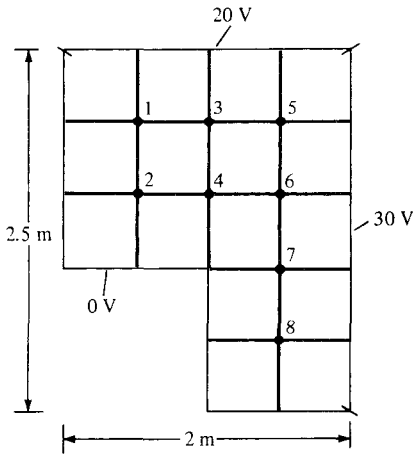


Figure 15.11 For Example 15.3.

Method 1 (Iteration Method): We first set the initial values of the potential at the free nodes equal to zero. We apply eq. (15.16) to each free node using the newest surrounding potentials each time the potential at that node is calculated. For the first iteration:

$$V_1 = 1/4(0 + 20 + 0 + 0) = 5$$

$$V_2 = 1/4(5 + 0 + 0 + 0) = 1.25$$

$$V_3 = 1/4(5 + 20 + 0 + 0) = 6.25$$

$$V_4 = 1/4(1.25 + 6.25 + 0 + 0) = 1.875$$

and so on. To avoid confusion, each time a new value at a free node is calculated, we cross out the old value as shown in Figure 15.12. After V_8 is calculated, we start the second iteration at node 1:

$$V_1 = 1/4(0 + 20 + 1.25 + 6.25) = 6.875$$

$$V_2 = 1/4(6.875 + 0 + 0 + 1.875) = 2.187$$

and so on. If this process is continued, we obtain the uncrossed values shown in Figure 15.12 after five iterations. After 10 iterations (not shown in Figure 15.12), we obtain

$$V_1 = 10.04, \quad V_2 = 4.956, \quad V_3 = 15.22, \quad V_4 = 9.786$$

$$V_5 = 21.05, \quad V_6 = 18.97, \quad V_7 = 15.06, \quad V_8 = 11.26$$

Method 2 (Band Matrix Method): This method reveals the sparse structure of the problem. We apply eq. (15.16) to each free node and keep the known terms (prescribed potentials at the fixed nodes) on the right side while the unknown terms (potentials at free nodes) are on the left side of the resulting system of simultaneous equations, which will be expressed in matrix form as $[A]\,[V] = [B]$.
For node 1,

$$-4V_1 + V_2 + V_3 = -20 - 0$$

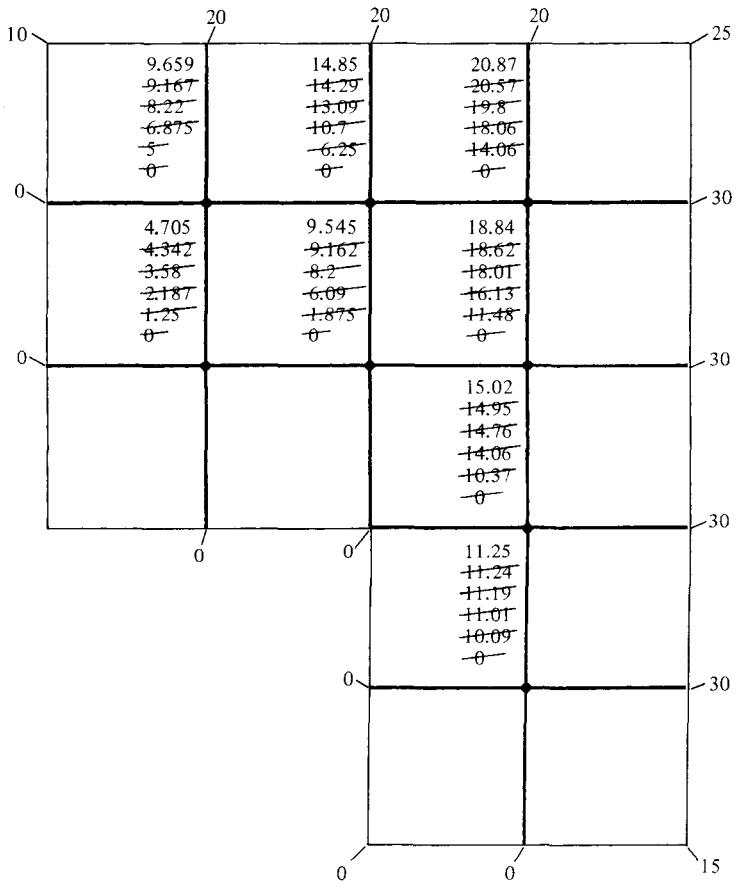


Figure 15.12 For Example 15.3; the values not crossed out are the solutions after five iterations.

For node 2,

$$V_1 + 4V_2 + V_4 = -0 - 0$$

For node 3,

$$V_1 - 4V_3 + V_4 + V_5 = -20$$

For node 4,

$$V_2 + V_3 - 4V_4 + V_6 = -0$$

For node 5,

$$V_3 - 4V_5 + V_6 = -20 - 30$$

For node 6,

$$V_4 + V_5 - 4V_6 + V_7 = -30$$

For node 7,

$$V_6 - 4V_7 + V_8 = -30 - 0$$

For node 8,

$$V_7 - 4V_8 = -0 - 0 - 30$$

Note that we have five terms at each node since we are using a five-node molecule. The eight equations obtained are put in matrix form as:

$$\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ -20 \\ 0 \\ -50 \\ -30 \\ -30 \\ -30 \end{bmatrix}$$

or

$$[A]\,[V] = [B]$$

where $[A]$ is the band, sparse matrix, $[V]$ is the column matrix consisting of the unknown potentials at the free nodes, and $[B]$ is the column matrix formed by the potential at the fixed nodes. The "band" nature of $[A]$ is shown by the dotted loop.

Notice that matrix $[A]$ could have been obtained directly from Figure 15.11 without writing down eq. (15.16) at each free node. To do this, we simply set the diagonal (or self) terms $A_{ii} = -4$ and set $A_{ij} = 1$ if i and j nodes are connected or $A_{ij} = 0$ if i and j nodes are not directly connected. For example, $A_{23} = A_{32} = 0$ because nodes 2 and 3 are not connected whereas $A_{46} = A_{64} = 1$ because nodes 4 and 6 are connected. Similarly, matrix $[B]$ is obtained directly from Figure 15.11 by setting B_i equal to minus the sum of the potentials at fixed nodes connected to node i. For example, $B_5 = -(20 + 30)$ because node 5 is connected to two fixed nodes with potentials 20 V and 30 V. If node i is not connected to any fixed node, $B_i = 0$.

By inverting matrix $[A]$ using Matlab, we obtain

$$[V] = [A]^{-1}\,[B]$$

or

$$V_1 = 10.04, \quad V_2 = 4.958, \quad V_3 = 15.22, \quad V_4 = 9.788$$
$$V_5 = 21.05, \quad V_6 = 18.97, \quad V_7 = 15.06, \quad V_8 = 11.26$$

which compares well with the result obtained using the iteration method.

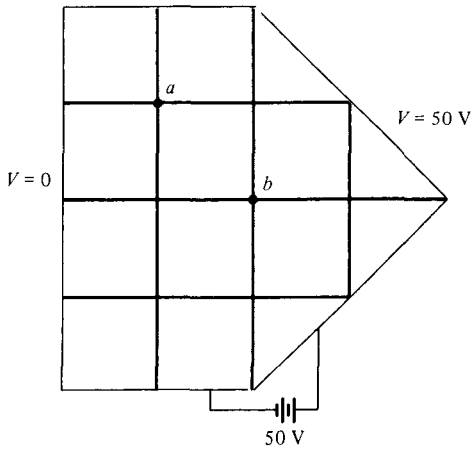


Figure 15.13 For Practice Exercise 15.3.

PRACTICE EXERCISE 15.3

Use the iteration method to find the finite difference approximation to the potentials at points a and b of the system in Figure 15.13.

Answer: $V_a = 10.01$ V, $V_b = 28.3$ V.

EXAMPLE 15.4

Obtain the solution of Laplace's equation for an infinitely long trough whose rectangular cross section is shown in Figure 15.14. Let $V_1 = 10$ V, $V_2 = 100$ V, $V_3 = 40$ V, and $V_4 = 0$ V.

Solution:

We shall solve this problem using the iteration method. In this case, the solution region has a regular boundary. We can easily write a program to determine the potentials at the grid points within the trough. We divide the region into square meshes. If we decide to use a 15×10 grid, the number of grid points along x is $15 + 1 = 16$ and the number of grid points along y is $10 + 1 = 11$. The mesh size $h = 1.5/15 = 0.1$ m. The 15×10 grid is il-

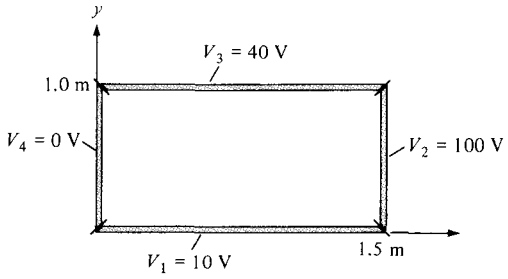


Figure 15.14 For Example 15.4.

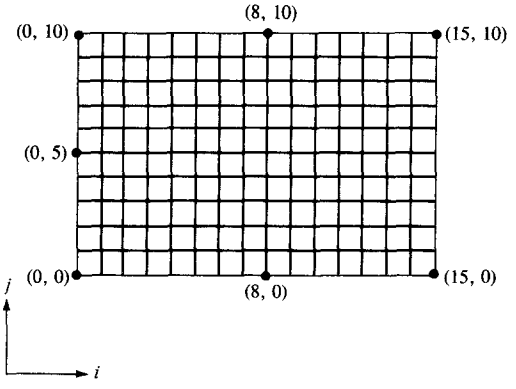


Figure 15.15 For Example 15.4; a 15 × 10 grid.

lustrated in Figure 15.15. The grid points are numbered (i, j) starting from the lower left-hand corner of the trough. Applying eq. (15.15) and using the iteration method, the computer program in Figure 15.16 was developed to determine the potential at the free nodes. At points $(x, y) = (0.5, 0.5)$, $(0.8, 0.8)$, $(1.0, 0.5)$, and $(0.8, 0.2)$ corresponding to $(i, j) = (5, 5)$, $(8, 8)$, $(10, 5)$, and $(8, 2)$, respectively, the potentials after 50, 100, and 200 iterations are shown in Table 15.1. The exact values (see Problem 6.18(c)), obtained using the method of separation of variables and a program similar to that of Figure 6.11, are also shown. It should be noted that the degree of accuracy depends on the mesh size h. It is always desirable to make h as small as possible. Also note that the potentials at the fixed nodes are held constant throughout the calculations.

```
%   USING FINITE DIFFERENCE (ITERATION) METHOD
%   THIS PROGRAM SOLVES THE TWO-DIMENSIONAL BOUNDARY-VALUE
%   PROBLEM (LAPLACE'S EQUATION) SHOWN IN FIG. 15.14.
%   ni = NO. OF ITERATIONS
%   nx = NO. OF X GRID POINTS
%   ny = NO. OF Y GRID POINTS
%   v(i,j) = POTENTIAL AT GRID POINT (i,j) OR (x,y) WITH
%   NODE NUMBERING STARTING FROM THE LOWER LEFT-HAND
%   CORNER OF THE TROUGH

v1 = 10.0;
v2 = 100.0;
v3 = 40.0;
v4 = 0.0;
ni = 200;
nx = 16;
ny = 11;
% SET INITIAL VALUES EQUAL TO ZEROES
v = zeros(nx,ny);
% FIX POTENTIALS ARE FIXED NODES
```

Figure 15.16 Computer Program for Example 15.4.

```
for i=2:nx-1
   v(i,1) = v1;
   v(i,ny) = v3;
end
for j=2:ny-1
   v(1,j) = v4;
   v(nx,j) = v2;
end
v(1,1) = 0.5*(v1 + v4);
v(nx,1) = 0.5*(v1 + v2);
v(1,ny) = 0.5*(v3 + v4);
v(nx,ny) = 0.5*(v2 + v3);
% NOW FIND v(i,j) USING EQ. (15.15) AFTER ni ITERATIONS
for k=1:ni
   for i=2:nx-1
      for j=2:ny-1
         v(i,j) = 0.25*( v(i+1,j) + v(i-1,j) + v(i,j+1) + v(1,j-1) );
      end
   end
end
diary a:test1.out
[v(6,6), v(9,9), v(11,6), v(9,3)]
[ [1:nx, 1:ny] v(i,j) ]
diary off
```

Figure 15.16 (Continued)

TABLE 15.1 Solution of Example 15.4 (Iteration Method) at Selected Points

Coordinates (x, y)	Number of Iterations			Exact Value
	50	100	200	
(0.5, 0.5)	20.91	22.44	22.49	22.44
(0.8, 0.8)	37.7	38.56	38.59	38.55
(1.0, 0.5)	41.83	43.18	43.2	43.22
(0.8, 0.2)	19.87	20.94	20.97	20.89

PRACTICE EXERCISE 15.4

Consider the trough of Figure 15.17. Use a five-node finite difference scheme to find the potential at the center of the trough using (a) a 4×8 grid, and (b) a 12×24 grid.

Answer: (a) 23.8 V, (b) 23.89 V.

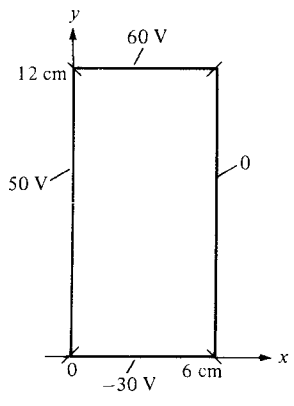


Figure 15.17 For Practice Exercise 15.4.

15.4 THE MOMENT METHOD

Like the finite difference method, the moment method[2] or the method of moments (MOM) has the advantage of being conceptually simple. While the finite difference method is used in solving differential equations, the moment method is commonly used in solving integral equations.

For example, suppose we want to apply the moment method to solve Poisson's equation in eq. (15.9a). It can be shown that an integral solution to Poisson's equation is

$$V = \int \frac{\rho_v \, dv}{4\pi\varepsilon r} \tag{15.26}$$

We recall from Chapter 4 that eq. (15.26) can be derived from Coulomb's law. We also recall that given the charge distribution $\rho_v(x, y, z)$, we can always find the potential $V(x, y, z)$, the electric field $\mathbf{E}(x, y, z)$, and the total charge Q. If, on the other hand, the potential V is known and the charge distribution is unknown, how do we determine ρ_v from eq. (15.26)? In that situation, eq. (15.26) becomes what is called an *integral equation.*

An **integral equation** is one involving the unknown function under the integral sign.

It has the general form of

$$V(x) = \int_a^b K(x, t)\, \rho(t)\, dt \tag{15.27}$$

where the functions $K(x, t)$ and $V(t)$ and the limits a and b are known. The unknown function $\rho(t)$ is to be determined; the function $K(x, t)$ is called the *kernel* of the equation. The

[2]The term "moment method" was first used in Western literature by Harrington. For further exposition on the method, see R. F. Harrington, *Field Computation by Moment Methods.* Malabar, FL: Krieger, 1968.

moment method is a common numerical technique used in solving integral equations such as in eq. (15.27). The method is probably best explained with an example.

Consider a thin conducting wire of radius a, length $L(L \gg a)$ located in free space as shown in Figure 15.18. Let the wire be maintained at a potential of V_o. Our goal is to determine the charge density ρ_L along the wire using the moment method. Once we determine ρ_L, related field quantities can be found. At any point on the wire, eq. (15.26) reduces to an integral equation of the form

$$V_o = \int_0^L \frac{\rho_L \, dl}{4\pi\varepsilon_o r} \tag{15.28}$$

Since eq. (15.28) applies for observation points everywhere on the wire, at a fixed point y_k known as the *match point.*

$$V_o = \frac{1}{4\pi\varepsilon_o} \int_0^L \frac{\rho_L(y)\, dy}{|y_k - y|} \tag{15.29}$$

We recall from calculus that integration is essentially finding the area under a curve. If Δy is small, the integration of $f(y)$ over $0 < y < L$ is given by

$$\begin{aligned} \int_0^L f(y)\, dy &\simeq f(y_1)\,\Delta y + f(y_2)\,\Delta y + \cdots + f(y_N)\,\Delta y \\ &= \sum_{k=1}^{N} f(y_k)\,\Delta y \end{aligned} \tag{15.30}$$

where the interval L has been divided into N units of each length Δy. With the wire divided into N segments of equal length Δ as shown in Figure 15.19, eq. (15.29) becomes

$$4\pi\varepsilon_o V_o \simeq \frac{\rho_1\,\Delta}{|y_k - y_1|} + \frac{\rho_2\,\Delta}{|y_k - y_2|} + \cdots + \frac{\rho_N\,\Delta}{|y_k - y_N|} \tag{15.31}$$

where $\Delta = L/N = \Delta y$. The assumption in eq. (15.31) is that the unknown charge density ρ_k on the kth segment is constant. Thus in eq. (15.31), we have unknown constants ρ_1,

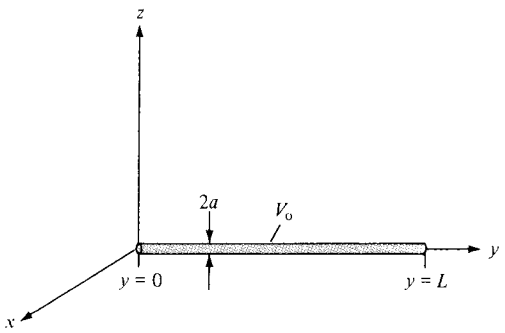


Figure 15.18 Thin conducting wire held at a constant potential.

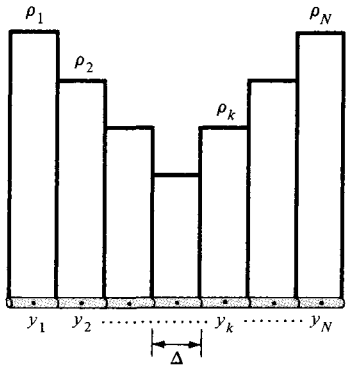


Figure 15.19 Division of the wire into N segments.

$\rho_2, \ldots, \rho_N$. Since eq. (15.31) must hold at all points on the wire, we obtain N similar equations by choosing N match points at $y_1, y_2, \ldots, y_k, \ldots y_N$ on the wire. Thus we obtain

$$4\pi\varepsilon_0 V_0 = \frac{\rho_1 \Delta}{|y_1 - y_1|} + \frac{\rho_2 \Delta}{|y_1 - y_2|} + \cdots + \frac{\rho_N \Delta}{|y_1 - y_N|} \tag{15.32a}$$

$$4\pi\varepsilon_0 V_0 = \frac{\rho_1 \Delta}{|y_2 - y_1|} + \frac{\rho_2 \Delta}{|y_2 - y_2|} + \cdots + \frac{\rho_N \Delta}{|y_2 - y_N|} \tag{15.32b}$$

$$\vdots$$

$$4\pi\varepsilon_0 V_0 = \frac{\rho_1 \Delta}{|y_N - y_1|} + \frac{\rho_2 \Delta}{|y_N - y_2|} + \cdots + \frac{\rho_N \Delta}{|y_N - y_N|} \tag{15.32c}$$

The idea of matching the left-hand side of eq. (15.29) with the right-hand side of the equation at the match points is similar to the concept of taking moments in mechanics. Here lies the reason this technique is called moment method. Notice from Figure 15.19 that the match points $y_1, y_2, \ldots, y_N$ are placed at the center of each segment. Equation (15.32) can be put in matrix form as

$$[B] = [A]\,[\rho] \tag{15.33}$$

where

$$[B] = 4\pi\varepsilon_0 V_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{15.34}$$

$$[A] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \tag{15.35a}$$

$$A_{mn} = \frac{\Delta}{|y_m - y_n|}, \qquad m \neq n \tag{15.35b}$$

$$[\rho] = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \rho_N \end{bmatrix} \tag{15.36}$$

In eq. (15.33), $[\rho]$ is the matrix whose elements are unknown. We can determine $[\rho]$ from eq. (15.33) using Cramer's rule, matrix inversion, or Gaussian elimination technique. Using matrix inversion,

$$\boxed{[\rho] = [A]^{-1}[B]} \tag{15.37}$$

where $[A]^{-1}$ is the inverse of matrix $[A]$. In evaluating the diagonal elements (or self terms) of matrix $[A]$ in eq. (15.32) or (15.35), caution must be exercised. Since the wire is conducting, a surface charge density ρ_S is expected over the wire surface. Hence at the center of each segment,

$$\begin{aligned} V(\text{center}) &= \frac{1}{4\pi\varepsilon_o} \int_0^{2\pi} \int_{-\Delta/2}^{\Delta/2} \frac{\rho_S a \, d\phi \, dy}{[a^2 + y^2]^{1/2}} \\ &= \frac{2\pi a \rho_S}{4\pi\varepsilon_o} \ln \left\{ \frac{\Delta/2 + [(\Delta/2)^2 + a^2]^{1/2}}{-\Delta/2 + [(\Delta/2)^2 + a^2]^{1/2}} \right\} \end{aligned}$$

Assuming $\Delta \gg a$,

$$\begin{aligned} V(\text{center}) &= \frac{2\pi a \rho_S}{4\pi\varepsilon_o} 2 \ln \left(\frac{\Delta}{a} \right) \\ &= \frac{2\rho_L}{4\pi\varepsilon_o} \ln \left(\frac{\Delta}{a} \right) \end{aligned} \tag{15.38}$$

where $\rho_L = 2\pi a \rho_S$. Thus, the self terms ($m = n$) are

$$A_{nn} = 2 \ln \left(\frac{\Delta}{a} \right) \tag{15.39}$$

Equation (15.33) now becomes

$$\begin{bmatrix} 2\ln\left(\frac{\Delta}{a}\right) & \frac{\Delta}{|y_1 - y_2|} & \cdots & \frac{\Delta}{|y_1 - y_N|} \\ \frac{\Delta}{|y_2 - y_1|} & 2\ln\left(\frac{\Delta}{a}\right) & \cdots & \frac{\Delta}{|y_2 - y_N|} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\Delta}{|y_N - y_1|} & \frac{\Delta}{|y_N - y_2|} & \cdots & 2\ln\left(\frac{\Delta}{a}\right) \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \rho_N \end{bmatrix} = 4\pi\varepsilon_o V_o \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{15.40}$$

Using eq. (15.37) with eq. (15.40) and letting $V_o = 1\ V$, $L = 1$ m, $a = 1$ mm, and $N = 10$ ($\Delta = L/N$), a Matlab code such as in Figure 15.20 can be developed. The program in Figure 15.20 is self-explanatory. It inverts matrix [A] and plots ρ_L against y. The plot is shown in Figure 15.21. The program also determines the total charge on the wire using

$$Q = \int \rho_L \, dl \tag{15.41}$$

which can be written in discrete form as

$$Q = \sum_{k=1}^{N} \rho_k \Delta \tag{15.42}$$

With the chosen parameters, the value of the total charge was found to be $Q = 8.536$ pC. If desired, the electric field at any point can be calculated using

$$\mathbf{E} = \int \frac{\rho_L \, dl}{4\pi\varepsilon_o R^2} \mathbf{a}_R \tag{15.43}$$

which can be written as

$$\mathbf{E} = \sum_{k=1}^{N} \frac{\rho_k \Delta \mathbf{R}}{4\pi\varepsilon_o R^3} \tag{15.44}$$

where $R = |\mathbf{R}|$ and

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_k = (x - x_k)\mathbf{a}_x + (y - y_k)\mathbf{a}_y + (z - z_k)\mathbf{a}_z$$

$\mathbf{r} = (x, y, z)$ is the position vector of the observation point, and $\mathbf{r}_k = (x_k, y_k, z_k)$ is that of the source point.

Notice that to obtain the charge distribution in Figure 15.21, we have taken $N = 10$. It should be expected that a smaller value of N would give a less accurate result and a larger value of N would yield a more accurate result. However, if N is too large, we may have the computation problem of inverting the square matrix [A]. The capacity of the computing facilities at our disposal can limit the accuracy of the numerical experiment.

```
%  THIS PROGRAM DETERMINES THE CHARGE DISTRIBUTION
%  ON A CONDUCTING THIN WIRE, OF RADIUS AA AND
%  LENGTH L, MAINTAINED AT VO VOLT
%  THE WIRE IS LOCATED AT 0 < Y < L
%  ALL DIMENSIONS ARE IN S.I. UNITS

%  MOMENT METHOD IS USED
%  N IS THE NO. OF SEGMENTS INTO WHICH THE WIRE IS DIVIDED
%  RHO IS THE LINE CHARGE DENSITY, RHO = INV(A)*B

%  FIRST, SPECIFY PROBLEM PARAMETERS
ER = 1.0;
EO = 8.8541e-12;
VO = 1.0;
AA = 0.001;
L = 1.0;
N = 20;
DELTA = L/N;
%  SECOND, CALCULATE THE ELEMENTS OF THE COEFFICIENT
%  MATRIX A
I=1:N;
Y=DELTA*(I-0.5);
for i=1:N
   for j=1:N
      if(i ~=j)
         A(i,j)=DELTA/abs(Y(i)-Y(j));
      else
         A(i,j)=2.0*log(DELTA/AA);
      end
   end
end
%  NOW DETERMINE THE MATRIX OF CONSTANT VECTOR B
%  AND FIND Q
B = 4.0*pi*EO*ER*VO*ones(N,1);
C = inv(A);
RHO = C*B;
SUM = 0.0;
for I=1:N
   SUM = SUM + RHO(I);
end
Q=SUM*DELTA;
diary  a:exam145a.out
[EO,Q]
[ [1:N]' Y' RHO ]
diary off
%  FINALLY PLOT RHO AGAINST Y
plot(Y,RHO)
xlabel('y (cm)'), ylabel('rho_L (pC/m)')
```

Figure 15.20 Matlab code for calculating the charge distribution on the wire in Figure 15.18.

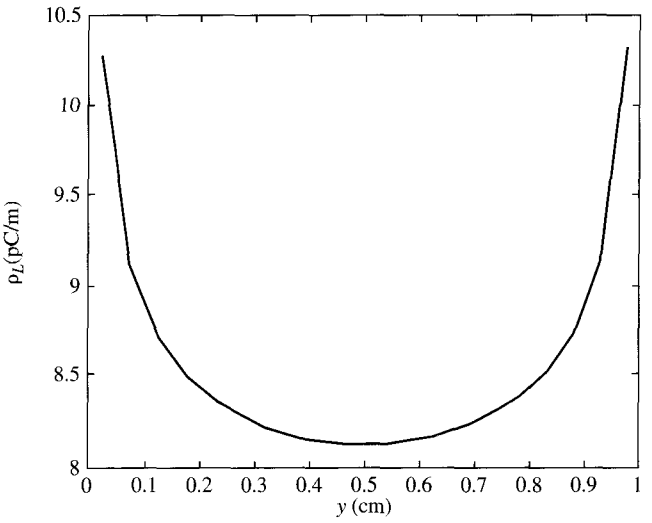


Figure 15.21 Plot of ρ_L against y.

EXAMPLE 15.5

Use the moment method to find the capacitance of the parallel-plane capacitor of Figure 15.22. Take $a = 1$ m, $b = 1$ m, $d = 1$ m, and $\varepsilon_r = 1.0$.

Solution:

Let the potential difference between the plates be $V_o = 2$ V so that the top plate P_1 is maintained at $+1$ V while the bottom plate P_2 is at -1 V. We would like to determine the surface charge density ρ_S on the plates so that the total charge on each plate can be found as

$$Q = \int \rho_S \, dS$$

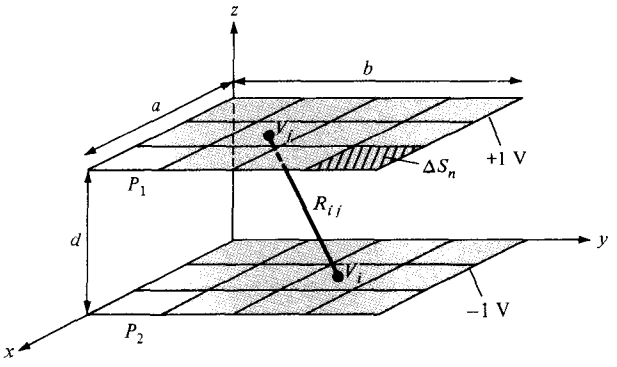


Figure 15.22 Parallel-plate capacitor; for Example 15.5.

Once Q is known, we can calculate the capacitance as

$$C = \frac{Q}{V_o} = \frac{Q}{2}$$

To determine ρ_S using the moment method, we divide P_1 into n subsections: ΔS_1, ΔS_2, . . ., ΔS_n and P_2 into n subsections: ΔS_{n+1}, ΔS_{n+2}, . . ., ΔS_{2n}. The potential V_i at the center of a typical subsection ΔS_i is

$$V_i = \int_S \frac{\rho_S \, dS}{4\pi\varepsilon_o R} \simeq \sum_{j=1}^{2n} \frac{1}{4\pi\varepsilon_o} \int_{\Delta S_j} \frac{\rho_j \, dS}{R_{ij}}$$
$$= \sum_{j=1}^{2n} \rho_j \frac{1}{4\pi\varepsilon_o} \int_{\Delta S_j} \frac{dS}{R_{ij}}$$

It has been assumed that there is uniform charge distribution on each subsection. The last equation can be written as

$$V_i = \sum_{j=1}^{2n} \rho_j A_{ij}$$

where

$$A_{ij} = \frac{1}{4\pi\varepsilon_o} \int_{\Delta S_j} \frac{dS}{R_{ij}}$$

Thus

$$V_1 = \sum_{j=1}^{2n} \rho_j A_{1j} = 1$$

$$V_2 = \sum_{j=1}^{2n} \rho_j A_{2j} = 1$$

$$\vdots$$

$$V_n = \sum_{j=1}^{2n} \rho_j A_{nj} = 1$$

$$V_{n+1} = \sum_{j=1}^{2n} \rho_j A_{n+1,j} = -1$$

$$\vdots$$

$$V_{2n} = \sum_{j=1}^{2n} \rho_j A_{2n,j} = -1$$

yielding a set of $2n$ simultaneous equations with $2n$ unknown charge densities ρ_j. In matrix form,

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,2n} \\ A_{21} & A_{22} & \cdots & A_{2,2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_{2n,1} & A_{2n,2} & \cdots & A_{2n,2n} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \rho_{2n} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \\ -1 \end{bmatrix}$$

or

$$[A]\,[\rho] = [B]$$

Hence,

$$[\rho] = [A]^{-1}\,[B]$$

where $[B]$ is the column matrix defining the potentials and $[A]$ is a square matrix containing elements A_{ij}. To determine A_{ij}, consider the two subsections i and j shown in Figure 15.23 where the subsections could be on different plates or on the same plate.

$$A_{ij} = \frac{1}{4\pi\varepsilon_o} \int_{y=y_1}^{y_2} \int_{x=x_1}^{x_2} \frac{dx\,dy}{R_{ij}}$$

where

$$R_{ij} = [(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2]^{1/2}$$

For the sake of convenience, if we assume that the subsections are squares,

$$x_2 - x_1 = \Delta\ell = y_2 - y_1$$

it can be shown that

$$A_{ij} = \frac{\Delta S_i}{4\pi\varepsilon_o R_{ij}} = \frac{(\Delta\ell)^2}{4\pi\varepsilon_o R_{ij}} \qquad i \neq j$$

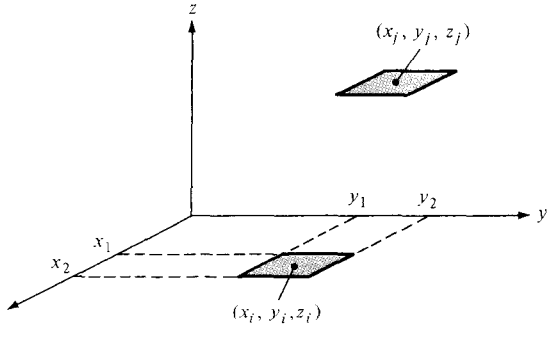


Figure 15.23 Subsections i and j; for Example 15.5.

and

$$A_{ii} = \frac{\Delta\ell}{\pi\varepsilon_o}\ln(1 + \sqrt{2}) = \frac{\Delta\ell}{\pi\varepsilon_o}(0.8814)$$

With these formulas, the Matlab code in Figure 15.24 was developed. With $n = 9$, $C = 26.51$ pF, with $n = 16$, $C = 27.27$ pF, and with $n = 25$, $C = 27.74$ pF.

```
%   USING THE METHOD OF MOMENT,
%   THIS PROGRAM DETERMINES THE CAPACITANCE OF A
%   PARALLEL-PLATE CAPACITOR CONSISTING OF TWO CONDUCTING
%   PLATES, EACH OF DIMENSION AA x BB, SEPARATED BY A
%   DISTANCE D, AND MAINTAINED AT 1 VOLT AND -1 VOLT

%   ONE PLATE IS LOCATED ON THE Z=0 PLANE WHILE THE OTHER
%   IS LOCATED ON THE Z=D PLANE

%   ALL DIMENSIONS ARE IN S.I. UNITS
%   N IS THE NUMBER IS SUBSECTIONS INTO WHICH EACH PLATE IS
DIVIDED

%   FIRST, SPECIFY THE PARAMETERS

ER = 1.0;
EO = 8.8541e-12;
AA = 1.0;
BB = 1.0;
D = 1.0;
N = 9;
NT = 2*N;
M = sqrt(N);
DX = AA/M;
DY = BB/M;
DL = DX;
%   SECOND, CALCULATE THE ELEMENTS OF THE COEFFICIENT
%   MATRIX A
K = 0;
for K1=1:2
   for K2=1:M
      for K3=1:M
         K = K + 1;
         X(K) = DX*(K2 - 0.5);
         Y(K) = DY*(K3 - 0.5);
      end
   end
end
```

Figure 15.24 Matlab program for Example 15.5.

```
for K1=1:N
   Z(K1) = 0.0;
   Z(K1+N) = D;
end
for I=1:NT
   for J=1:NT
      if(I==J)
         A(I,J) = DL*0.8814/(pi*EO);
      else
       R = sqrt( (X(I)-X(J))^2 = (Y(I)-Y(J))^2 + (Z(I)-Z(J))^2 );
         A(I,J) = DL^2/(4.*pi*EO*R);
      end
   end
end
%  NOW DETERMINE THE MATRIX OF CONSTANT VECTOR B
for K=1:N
   B(K) = 1.0;
   B(K+N) = -1.0;
end
%  INVERT A AND CALCULATE RHO CONSISTING
%  THE UNKNOWN ELEMENTS
%  ALSO CALCULATE THE TOTAL CHARGE Q AND CAPACITANCE C
F = inv(A);
RHO = F*B';
SUM = 0.0;
for I=1:N
   SUM = SUM + RHO(I);
end
Q = SUM*(DL^2);
VO = 2.0;
C = abs(Q)/VO;
diary a:exam145b.out
[C]
[ [1:NT]'  X   Y'  Z'  RHO ]
diary off
```

Figure 15.24 (Continued)

PRACTICE EXERCISE 15.5

Using the moment method, write a program to determine the capacitance of two identical parallel conducting wires separated at a distance y_o and displaced by x_o as shown in Figure 15.25. If each wire is of length L and radius a, find the capacitance for cases $x_o = 0, 0.2, 0.4, \ldots, 1.0$ m. Take $y_o = 0.5$ m, $L = 1$ m, $a = 1$ mm, $\varepsilon_r = 1$.

Answer: For $N = 10$ = number of segments per wire, see Table 15.2.

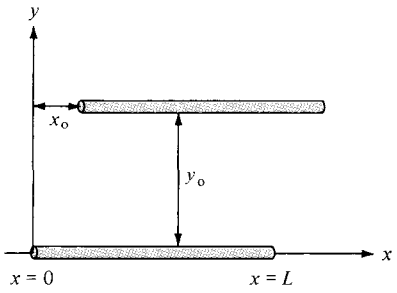


Figure 15.25 Parallel conducting wires of Practice Exercise 15.5.

TABLE 15.2 Capacitance for Practice Exercise 15.5

x_o (m)	C (pF)
0.0	4.91
0.2	4.891
0.4	4.853
0.6	4.789
0.8	4.71
1.0	4.643

15.5 THE FINITE ELEMENT METHOD

The finite element method (FEM) has its origin in the field of structural analysis. The method was not applied to EM problems until 1968.[3] Like the finite difference method, the finite element method is useful in solving differential equations. As noticed in Section 15.3, the finite difference method represents the solution region by an array of grid points; its application becomes difficult with problems having irregularly shaped boundaries. Such problems can be handled more easily using the finite element method.

The finite element analysis of any problem involves basically four steps: (a) discretizing the solution region into a finite number of subregions or *elements,* (b) deriving governing equations for a typical element, (c) assembling of all elements in the solution region, and (d) solving the system of equations obtained.

A. Finite Element Discretization

We divide the solution region into a number of *finite elements* as illustrated in Figure 15.26 where the region is subdivided into four nonoverlapping elements (two triangular and two quadrilateral) and seven nodes. We seek an approximation for the potential V_e within an

[3]See P. P. Silvester and R. L. Ferrari, *Finite Elements for Electrical Engineers.* Cambridge, England: Cambridge Univ. Press, 1983.

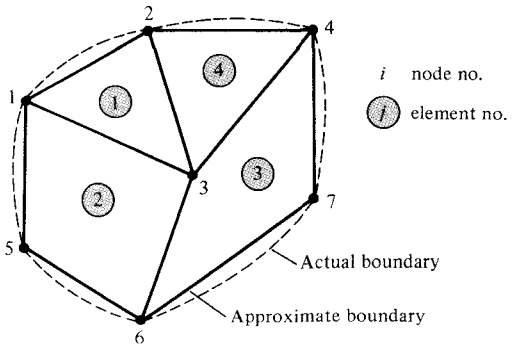


Figure 15.26 A typical finite element subdivision of an irregular domain.

element e and then interrelate the potential distributions in various elements such that the potential is continuous across interelement boundaries. The approximate solution for the whole region is

$$V(x, y) \simeq \sum_{e=1}^{N} V_e(x, y) \tag{15.45}$$

where N is the number of triangular elements into which the solution region is divided.

The most common form of approximation for V_e within an element is polynomial approximation, namely

$$V_e(x, y) = a + bx + cy \tag{15.46}$$

for a triangular element and

$$V_e(x, y) = a + bx + cy + dxy \tag{15.47}$$

for a quadrilateral element. The potential V_e in general is nonzero within element e but zero outside e. It is difficult to approximate the boundary of the solution region with quadrilateral elements; such elements are useful for problems whose boundaries are sufficiently regular. In view of this, we prefer to use triangular elements throughout our analysis in this section. Notice that our assumption of linear variation of potential within the triangular element as in eq. (15.46) is the same as assuming that the electric field is uniform within the element; that is,

$$\mathbf{E}_e = -\nabla V_e = -(b\mathbf{a}_x + c\mathbf{a}_y) \tag{15.48}$$

B. Element Governing Equations

Consider a typical triangular element shown in Figure 15.27. The potential V_{e1}, V_{e2}, and V_{e3} at nodes 1, 2, and 3, respectively, are obtained using eq. (15.46); that is,

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{15.49}$$

The coefficients a, b, and c are determined from eq. (14.49) as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \tag{15.50}$$

Substituting this into eq. (15.46) gives

$$V_e = [1 \quad x \quad y] \frac{1}{2A} \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

or

$$\boxed{V_e = \sum_{i=1}^{3} \alpha_i(x, y)\, V_{ei}} \tag{15.51}$$

where

$$\alpha_1 = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)\, x + (x_3 - x_2)\, y] \tag{15.52a}$$

$$\alpha_2 = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)\, x + (x_1 - x_3)\, y] \tag{15.52b}$$

$$\alpha_3 = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)\, x + (x_2 - x_1)\, y] \tag{15.52c}$$

and A is the area of the element e; that is,

$$\begin{aligned} 2A &= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \\ &= (x_1y_2 - x_2y_1) + (x_3y_1 - x_1y_3) + (x_2y_3 - x_3y_2) \end{aligned}$$

or

$$A = 1/2\,[x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] \tag{15.53}$$

The value of A is positive if the nodes are numbered counterclockwise (starting from any node) as shown by the arrow in Figure 15.27. Note that eq. (15.51) gives the potential at any point (x, y) within the element provided that the potentials at the vertices are known. This is unlike the situation in finite difference analysis where the potential is known at the grid points only. Also note that α_i are linear interpolation functions. They are called the *element shape functions* and they have the following properties:

$$\alpha_i(x_j, y_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{15.54a}$$

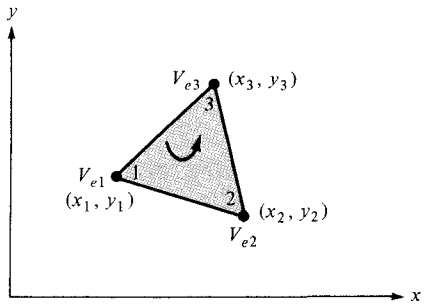


Figure 15.27 Typical triangular element; the local node numbering 1-2-3 must be counterclockwise as indicated by the arrow.

$$\sum_{i=1}^{3} \alpha_i(x, y) = 1 \tag{15.54b}$$

The shape functions α_1 and α_2, for example, are illustrated in Figure 15.28.

The energy per unit length associated with the element e is given by eq. (4.96); that is

$$W_e = \frac{1}{2} \int \varepsilon \, |\mathbf{E}|^2 \, dS = \frac{1}{2} \int \varepsilon \, |\nabla V_e|^2 \, dS \tag{15.55}$$

where a two-dimensional solution region free of charge ($\rho_S = 0$) is assumed. But from eq. (15.51),

$$\nabla V_e = \sum_{i=1}^{3} V_{ei} \, \nabla \alpha_i \tag{15.56}$$

Substituting eq. (15.56) into eq. (15.55) gives

$$W_e = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \varepsilon V_{ei} \left[\int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \right] V_{ej} \tag{15.57}$$

If we define the term in brackets as

$$C_{ij}^{(e)} = \int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \tag{15.58}$$

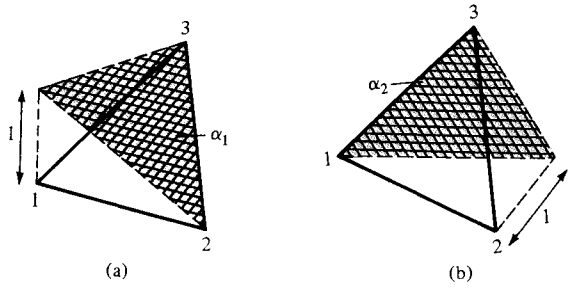


Figure 15.28 Shape functions α_1 and α_2 for a triangular element.

we may write eq. (15.57) in matrix form as

$$W_e = \frac{1}{2}\varepsilon\,[V_e]^T\,[C^{(e)}]\,[V_e] \tag{15.59}$$

where the superscript T denotes the transpose of the matrix,

$$[V_e] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \tag{15.60a}$$

and

$$[C^{(e)}] = \begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix} \tag{15.60b}$$

The matrix $[C^{(e)}$ is usually called the *element coefficient matrix.* The matrix element $C_{ij}^{(e)}$ of the coefficient matrix may be regarded as the coupling between nodes i and j; its value is obtained from eqs. (15.52) and (15.58). For example,

$$\begin{aligned} C_{12}^{(e)} &= \int \nabla\alpha_1 \cdot \nabla\alpha_2\, dS \\ &= \frac{1}{4A^2}\left[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)\right] \int dS \\ &= \frac{1}{4A}\left[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)\right] \end{aligned} \tag{15.61a}$$

Similarly:

$$C_{11}^{(e)} = \frac{1}{4A}\left[(y_2 - y_3)^2 + (x_3 - x_2)^2\right] \tag{15.61b}$$

$$C_{13}^{(e)} = \frac{1}{4A}\left[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)\right] \tag{15.61c}$$

$$C_{22}^{(e)} = \frac{1}{4A}\left[(y_3 - y_1)^2 + (x_1 - x_3)^2\right] \tag{15.61d}$$

$$C_{23}^{(e)} = \frac{1}{4A}\left[(y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1)\right] \tag{15.61e}$$

$$C_{33}^{(e)} = \frac{1}{4A}\left[(y_1 - y_2)^2 + (x_2 - x_1)^2\right] \tag{15.61f}$$

Also

$$C_{21}^{(e)} = C_{12}^{(e)}, \qquad C_{31}^{(e)} = C_{13}^{(e)}, \qquad C_{32}^{(e)} = C_{23}^{(e)} \tag{15.61g}$$

However, our calculations will be easier if we define

$$P_1 = (y_2 - y_3), \qquad P_2 = (y_3 - y_1), \qquad P_3 = (y_1 - y_2) \tag{15.62a}$$
$$Q_1 = (x_3 - x_2), \qquad Q_2 = (x_1 - x_3), \qquad Q_3 = (x_2 - x_1)$$

With P_i and Q_i ($i = 1, 2, 3$ are the local node numbers), each term in the element coefficient matrix is found as

$$\boxed{C_{ij}^{(e)} = \frac{1}{4A}[P_iP_j + Q_iQ_j]} \tag{15.62b}$$

where

$$\boxed{A = \frac{1}{2}(P_2Q_3 - P_3Q_2)} \tag{15.62c}$$

Note that $P_1 + P_2 + P_3 = 0 = Q_1 + Q_2 + Q_3$ and hence $\sum_{i=1}^{3} C_{ij}^{(e)} = 0 = \sum_{j=1}^{3} C_{ij}^{(e)}$. This may be used in checking our calculations.

C. Assembling of All Elements

Having considered a typical element, the next step is to assemble all such elements in the solution region. The energy associated with the assemblage of all elements in the mesh is

$$W = \sum_{e=1}^{N} W_e = \frac{1}{2}\varepsilon\,[V]^T\,[C]\,[V] \tag{15.63}$$

where

$$[V] = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{bmatrix} \tag{15.64}$$

n is the number of nodes, N is the number of elements, and $[C]$ is called the *overall* or *global coefficient matrix,* which is the assemblage of individual element coefficient matrices. The major problem now is obtaining $[C]$ from $[C^{(e)}]$.

The process by which individual element coefficient matrices are assembled to obtain the global coefficient matrix is best illustrated with an example. Consider the finite element mesh consisting of three finite elements as shown in Figure 15.29. Observe the numberings of the nodes. The numbering of nodes as 1, 2, 3, 4, and 5 is called *global* numbering. The numbering *i-j-k* is called *local* numbering and it corresponds with 1-2-3 of the element in

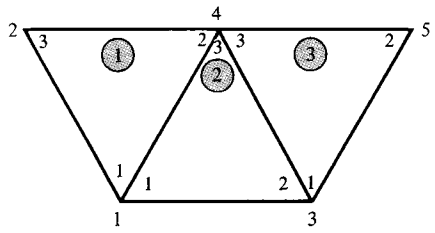


Figure 15.29 Assembly of three elements: *i-j-k* corresponds to local numbering 1-2-3 of the element in Figure 15.27.

Figure 15.27. For example, for element 3 in Figure 15.29, the global numbering 3-5-4 corresponds with local numbering 1-2-3 of the element in Figure 15.27. Note that the local numbering must be in counterclockwise sequence starting from any node of the element. For element 3, for example, we could choose 4-3-5 or 5-4-3 instead of 3-5-4 to correspond with 1-2-3 of the element in Figure 15.27. Thus the numbering in Figure 15.29 is not unique. However, we obtain the same $[C]$ whichever numbering is used. Assuming the particular numbering in Figure 15.29, the global coefficient matrix is expected to have the form

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix} \tag{15.65}$$

which is a 5×5 matrix since five nodes ($n = 5$) are involved. Again, C_{ij} is the coupling between nodes i and j. We obtain C_{ij} by utilizing the fact that the potential distribution must be continuous across interelement boundaries. The contribution to the i, j position in $[C]$ comes from all elements containing nodes i and j. To find C_{11}, for example, we observe from Figure 15.29 that global node 1 belongs to elements 1 and 2 and it is local node 1 in both; hence,

$$C_{11} = C_{11}^{(1)} + C_{11}^{(2)} \tag{15.66a}$$

For C_{22}, global node 2 belongs to element 1 only and is the same as local node 3; hence,

$$C_{22} = C_{33}^{(1)} \tag{15.66b}$$

For C_{44}, global node 4 is the same as local nodes 2, 3, and 3 in elements 1, 2, and 3, respectively; hence,

$$C_{44} = C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} \tag{15.66c}$$

For C_{14}, global link 14 is the same as the local links 12 and 13 in elements 1 and 2, respectively; hence,

$$C_{14} = C_{12}^{(1)} + C_{13}^{(2)} \tag{15.66d}$$

Since there is no coupling (or direct link) between nodes 2 and 3,

$$C_{23} = C_{32} = 0 \tag{15.66e}$$

Continuing in this manner, we obtain all the terms in the global coefficient matrix by inspection of Figure 15.29 as

$$[C] = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & 0 \\ C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} & 0 \\ C_{21}^{(2)} & 0 & C_{22}^{(2)} + C_{11}^{(3)} & C_{23}^{(2)} + C_{13}^{(3)} & C_{12}^{(3)} \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)} + C_{31}^{(3)} & C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} & C_{32}^{(3)} \\ 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} & C_{22}^{(3)} \end{bmatrix} \tag{15.67}$$

Note that element coefficient matrices overlap at nodes shared by elements and that there are 27 terms (nine for each of the three elements) in the global coefficient matrix $[C]$. Also note the following properties of the matrix $[C]$:

1. It is symmetric ($C_{ij} = C_{ji}$) just as the element coefficient matrix.
2. Since $C_{ij} = 0$ if no coupling exists between nodes i and j, it is evident that for a large number of elements $[C]$ becomes sparse and banded.
3. It is singular. Although this is not so obvious, it can be shown using the element coefficient matrix of eq. (15.60b).

D. Solving the Resulting Equations

From variational calculus, it is known that Laplace's (or Poisson's) equation is satisfied when the total energy in the solution region is minimum. Thus we require that the partial derivatives of W with respect to each nodal value of the potential be zero; that is,

$$\frac{\partial W}{\partial V_1} = \frac{\partial W}{\partial V_2} = \cdots = \frac{\partial W}{\partial V_n} = 0$$

or

$$\frac{\partial W}{\partial V_k} = 0, \qquad k = 1, 2, \ldots, n \tag{15.68}$$

For example, to get $\partial W/\partial V_1 = 0$ for the finite element mesh of Figure 15.29, we substitute eq. (15.65) into eq. (15.63) and take the partial derivative of W with respect to V_1. We obtain

$$\begin{aligned} 0 = \frac{\partial W}{\partial V_1} &= 2V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15} \\ &\quad + V_2C_{21} + V_3C_{31} + V_4C_{41} + V_5C_{51} \end{aligned}$$

or

$$0 = V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15} \tag{15.69}$$

In general, $\partial W/\partial V_k = 0$ leads to

$$0 = \sum_{i=1}^{n} V_i C_{ik} \tag{15.70}$$

where n is the number of nodes in the mesh. By writing eq. (15.70) for all nodes $k = 1, 2, \ldots, n$, we obtain a set of simultaneous equations from which the solution of $[V]^T = [V_1, V_2, \ldots, V_n]$ can be found. This can be done in two ways similar to those used in solving finite difference equations obtained from Laplace's (or Poisson's) equation.

Iteration Method:

This approach is similar to that used in finite difference method. Let us assume that node 1 in Figure 15.29, for example, is a free node. The potential at node 1 can be obtained from eq. (15.69) as

$$V_1 = -\frac{1}{C_{11}} \sum_{i=2}^{5} V_i C_{1i} \tag{15.71}$$

In general, the potential at a free node k is obtained from eq. (15.70) as

$$\boxed{V_k = -\frac{1}{C_{kk}} \sum_{i=1, i\neq k}^{n} V_i C_{ik}} \tag{15.72}$$

This is applied iteratively to all the free nodes in the mesh with n nodes. Since $C_{ki} = 0$ if node k is not directly connected to node i, only nodes that are directly linked to node k contribute to V_k in eq. (15.72).

Thus if the potentials at nodes connected to node k are known, we can determine V_k using eq. (15.72). The iteration process begins by setting the potentials at the free nodes equal to zero or to the average potential.

$$V_{\text{ave}} = 1/2\,(V_{\min} + V_{\max}) \tag{15.73}$$

where $V_{\min}$ and $V_{\max}$ are the minimum and maximum values of the prescribed potentials at the fixed nodes. With those initial values, the potentials at the free nodes are calculated using eq. (15.72). At the end of the first iteration, when the new values have been calculated for all the free nodes, the values become the old values for the second iteration. The procedure is repeated until the change between subsequent iterations becomes negligible.

Band Matrix Method:

If all free nodes are numbered first and the fixed nodes last, eq. (15.63) can be written such that

$$W = \frac{1}{2}\varepsilon \begin{bmatrix} V_f & V_p \end{bmatrix} \begin{bmatrix} C_{ff} & C_{fp} \\ C_{pf} & C_{pp} \end{bmatrix} \begin{bmatrix} V_f \\ V_p \end{bmatrix} \tag{15.74}$$

where subscripts f and p, respectively, refer to nodes with free and fixed (or prescribed) potentials. Since V_p is constant (it consists of known, fixed values), we only differentiate with respect to V_f so that applying eq. (15.68) to eq. (15.74) yields

$$C_{ff}V_p + C_{fp}V_p = 0$$

or

$$[C_{ff}]\,[V_f] = -[C_{fp}]\,[V_p] \tag{15.75}$$

This equation can be written as

$$[A]\,[V] = [B] \tag{15.76a}$$

or

$$\boxed{[V] = [A]^{-1}\,[B]} \tag{15.76b}$$

where $[V] = [V_f]$, $[A] = [C_{ff}]$, and $[B] = -[C_{fp}]\,[V_p]$. Since $[A]$ is, in general, nonsingular, the potential at the free nodes can be found using eq. (15.75). We can solve for $[V]$ in eq. (15.76a) using Gaussian elimination technique. We can also solve for $[V]$ in eq. (15.76b) using matrix inversion if the size of the matrix to be inverted is not large.

Notice that as from eq. (15.55) onward, our solution has been restricted to a two-dimensional problem involving Laplace's equation, $\nabla^2 V = 0$. The basic concepts developed in this section can be extended to finite element analysis of problems involving Poisson's equation ($\nabla^2 V = -\rho_v/\varepsilon$, $\nabla^2 \mathbf{A} = -\mu\mathbf{J}$) or wave equation ($\nabla^2\phi - \gamma^2\phi = 0$). A major problem associated with finite element analysis is the relatively large amount of computer memory required in storing the matrix elements and the associated computational time. However, several algorithms have been developed to alleviate the problem to some degree.

The finite element method (FEM) has a number of advantages over the finite difference method (FDM) and the method of moments (MOM). First, the FEM can easily handle complex solution region. Second, the generality of FEM makes it possible to construct a general-purpose program for solving a wide range of problems. A single program can be used to solve different problems (described by the same partial differential equations) with different solution regions and different boundary conditions; only the input data to the problem need be changed. However, FEM has its own drawbacks. It is harder to understand and program than FDM and MOM. It also requires preparing input data, a process that could be tedious.

EXAMPLE 15.6 Consider the two-element mesh shown in Figure 15.30(a). Using the finite element method, determine the potentials within the mesh.

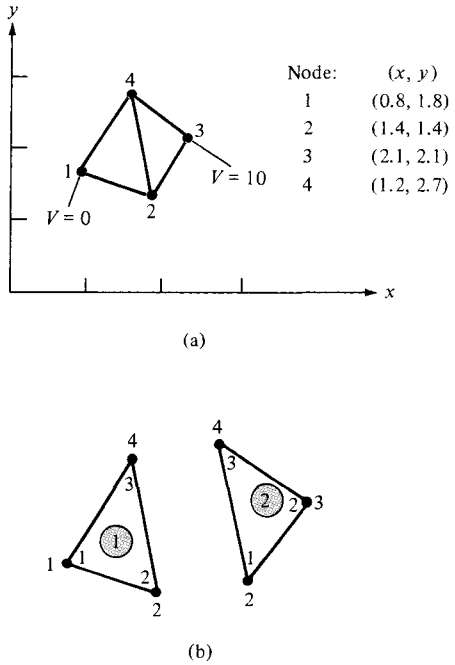


Figure 15.30 For Example 15.6: **(a)** two-element mesh, **(b)** local and global numbering of the elements.

Solution:

The element coefficient matrices can be calculated using eq. (15.62). For element 1, consisting of nodes 1-2-4 corresponding to the local numbering 1-2-3 as in Figure 15.30(b),

$$P_1 = -1.3, \qquad P_2 = 0.9, \qquad P_3 = 0.4$$

$$Q_1 = -0.2, \qquad Q_2 = -0.4, \qquad Q_3 = 0.6$$

$$A = 1/2\,(0.54 + 0.16) = 0.35$$

Substituting all these into eq. (15.62b) gives

$$[C^{(1)}] = \begin{bmatrix} 1.236 & -0.7786 & -0.4571 \\ -0.7786 & 0.6929 & 0.0857 \\ -0.4571 & 0.0857 & 0.3714 \end{bmatrix} \tag{15.6.1}$$

Similarly, for element 2 consisting of nodes 2-3-4 corresponding to local numbering 1-2-3 as in Figure 15.30(b),

$$P_1 = -0.6, \qquad P_2 = 1.3, \qquad P_3 = -0.7$$

$$Q_1 = -0.9, \qquad Q_2 = 0.2, \qquad Q_3 = 0.7$$

$$A = 1/2\,(0.91 + 0.14) = 0.525$$

Hence,

$$[C^{(2)}] = \begin{bmatrix} 0.5571 & -0.4571 & -0.1 \\ -0.4571 & 0.8238 & -0.3667 \\ -0.1 & -0.3667 & 0.4667 \end{bmatrix} \tag{15.6.2.}$$

Applying eq. (15.75) gives

$$\begin{bmatrix} C_{22} & C_{24} \\ C_{42} & C_{44} \end{bmatrix} \begin{bmatrix} V_2 \\ V_4 \end{bmatrix} = -\begin{bmatrix} C_{21} & C_{23} \\ C_{41} & C_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} \tag{15.6.3}$$

This can be written in a more convenient form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{22} & 0 & C_{24} \\ 0 & 0 & 1 & 0 \\ 0 & C_{42} & 0 & C_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -C_{21} & -C_{23} \\ 0 & 1 \\ -C_{41} & -C_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} \tag{15.6.4a}$$

or

$$[C]\,[V] = [B] \tag{15.6.4b}$$

The terms of the global coefficient matrix are obtained as follows:

$$C_{22} = C_{22}^{(1)} + C_{11}^{(2)} = 0.6929 + 0.5571 = 1.25$$

$$C_{42} = C_{24} = C_{23}^{(1)} + C_{13}^{(2)} = 0.0857 - 0.1 = -0.0143$$

$$C_{44} = C_{33}^{(1)} + C_{33}^{(2)} = 0.3714 + 0.4667 = 0.8381$$

$$C_{21} = C_{21}^{(1)} = -0.7786$$

$$C_{23} = C_{12}^{(2)} = -0.4571$$

$$C_{41} = C_{31}^{(1)} = -0.4571$$

$$C_{43} = C_{32}^{(2)} = -0.3667$$

Note that we follow local numbering for the element coefficient matrix and global numbering for the global coefficient matrix. Thus the square matrix $[C]$ is obtained as

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & -0.0143 \\ 0 & 0 & 1 & 0 \\ 0 & -0.0143 & 0 & 0.8381 \end{bmatrix} \tag{15.6.5}$$

and the matrix $[B]$ on the right-hand side of eq. (15.6.4a) is obtained as

$$[B] = \begin{bmatrix} 0 \\ 4.571 \\ 10.0 \\ 3.667 \end{bmatrix} \tag{15.6.6}$$

By inverting matrix $[C]$ in eq. (15.6.5), we obtain

$$[V] = [C]^{-1}[B] = \begin{bmatrix} 0 \\ 3.708 \\ 10.0 \\ 4.438 \end{bmatrix}$$

Thus $V_1 = 0$, $V_2 = 3.708$, $V_3 = 10$, and $V_4 = 4.438$. Once the values of the potentials at the nodes are known, the potential at any point within the mesh can be determined using eq. (15.51).

PRACTICE EXERCISE 15.6

Calculate the global coefficient matrix for the two-element mesh shown in Figure 15.31 when: (a) node 1 is linked with node 3 and the local numbering $(i - j - k)$ is as indicated in Figure 15.31(a), (b) node 2 is linked with node 4 with local numbering as in Figure 15.31(b).

Answer: (a)
$$\begin{bmatrix} 0.9964 & 0.05 & -0.2464 & -0.8 \\ 0.05 & 0.7 & -0.75 & 0.0 \\ -0.2464 & -0.75 & 1.5964 & -0.6 \\ -0.8 & 0.0 & -0.6 & 1.4 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1.333 & -0.7777 & 0.0 & -1.056 \\ -0.0777 & 0.8192 & -0.98 & 0.2386 \\ 0.0 & -0.98 & 2.04 & -1.06 \\ -1.056 & 0.2386 & -1.06 & 1.877 \end{bmatrix}$$

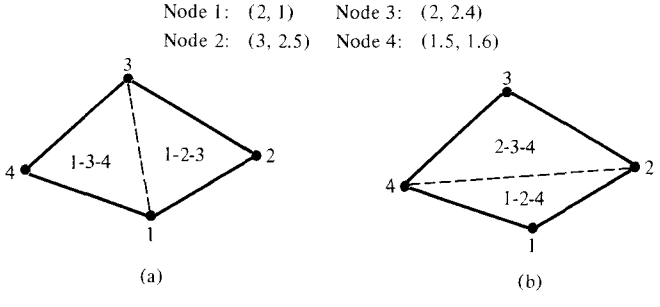


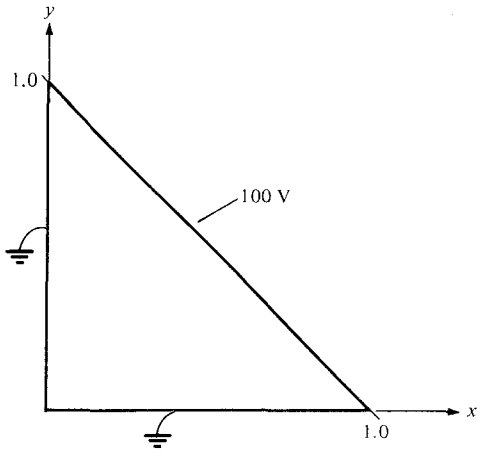
Figure 15.31 For Practice Exercise 15.6.

EXAMPLE 15.7

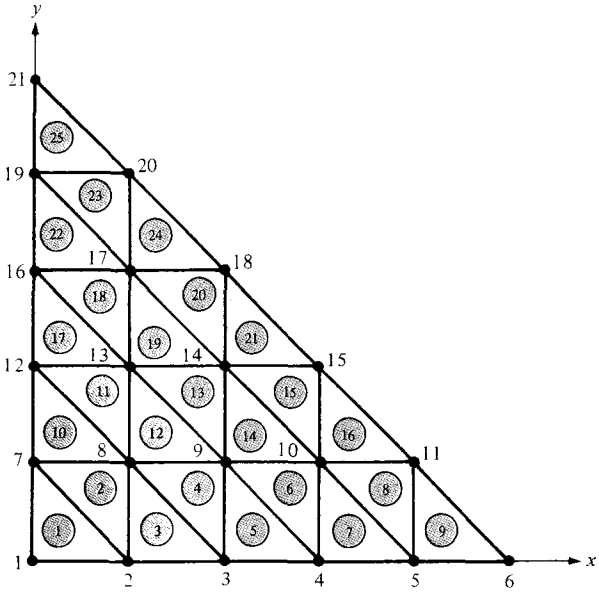
Write a program to solve Laplace's equation using the finite element method. Apply the program to the two-dimensional problem shown in Figure 15.32(a).

Solution:

The solution region is divided into 25 three-node triangular elements with the total number of nodes being 21 as shown in Figure 15.32(b). This is a necessary step in order to have input data defining the geometry of the problem. Based on our discussions in Section 15.5, a general Matlab program for solving problems involving Laplace's equation using three-node triangular elements was developed as in Figure 15.33. The devel-



(a)



(b)

Figure 15.32 For Example 15.7: **(a)** two-dimensional electrostatic problem, **(b)** solution region divided into 25 triangular elements.

```
%  FINITE ELEMENT SOLUTION OF LAPLACE'S EQUATION FOR
%  TWO-DIMENSIONAL PROBLEMS
%  TRIANGULAR ELEMENTS ARE USED
%  ND = NO. OF NODES
%  NE = NO. OF ELEMENTS
%  NP = NO. OF FIXED NODES (WHERE POTENTIAL IS PRESCRIBED)
%  NDP(I) = NODE NO. OF PRESCRIBED POTENTIAL, I=1,2,...,NP
%  VAL(I) = VALUE OF PRESCRIBED POTENTIAL AT NODE NDP(I)
%  NL(I,J) = LIST OF NODES FOR EACH ELEMENT I, WHERE
%            J=1,2,3 REFERS TO THE LOCAL NODE NUMBER
%  CE(I,J) = ELEMENT COEFFICIENT MATRIX
%  C(I,J) = GLOBAL COEFFICIENT MATRIX
%  B(I) = RIGHT-HAND SIDE MATRIX IN THE SYSTEM OF
%  SIMULTANEOUS EQUATIONS; SEE EQ. (15.6.4)
%  X(I), Y(I) = GLOBAL COORDINATES OF NODE I
%  XL(J), YL(J) = LOCAL COORDINATES OF NODE J=1,2,3
%  V(I) = POTENTIAL AT NODE I
%  MATRICES P(I) AND Q (I) ARE DEFINED IN EQ. (15.62a)

%  ******************************************************
%  FIRST STEP - INPUT DATA DEFINING GEOMETRY AND
                BOUNDARY CONDITIONS
%  ******************************************************

clear
input('Name of input data file = ')

% ********************************************************
%  SECOND STEP - EVALUATE COEFFICIENT MATRIX FOR EACH
                 ELEMENT AND ASSEMBLE GLOBALLY
% ********************************************************
B = zeros(ND,1);
C = zeros(ND,ND);
for I=1:NE
% FIND LOCAL COORDINATES XL(J), YL(J) FOR ELEMENT I
   K = NL(I,[1:3]);
   XL = X(K);
   YL = Y(K);
P=zeros(3,1);
Q=zeros(3,1);
   P(1) = YL(2) - YL(3);
   P(2) = YL(3) - YL(1);
   P(3) = YL(1) - YL(2);
   Q(1) = XL(3) - XL(2);
   Q(2) = XL(1) - XL(3);
   Q(3) = XL(2) - XL(1);
   AREA = 0.5*abs( P(2)*Q(3) - Q(2)*P(3) );
```

Figure 15.33 Computer program for Example 15.7.

```
%  DETERMINE COEFFICIENT MATRIX FOR ELEMENT I
   CE=(P*P'+Q*Q')/(4.0*AREA);
%  ASSEMBLE GLOBALLY - FIND C(I,J) AND B(I)
   for J=1:3
      IR = NL(I,J);
      IFLAG1=0;
%  CHECK IF ROW CORRESPONDS TO A FIXED NODE
   for K = 1:NP
      if (IR == NDP(K))
         C(IR,IR) = 1.0;
         B(IR) = VAL(K);
         IFLAG1=1;
      end
   end % end for K = 1:NP
   if(IFLAG1 == 0)
   for L = 1:3
     IC = NL(I,L);
     IFLAG2=0;
%  CHECK IF COLUMN CORRESPONDS TO A FIXED NODE
     for K=1:NP
        if ( IC == NDP(K) ),
          B(IR) = B(IR) - CE(J,L)*VAL(K);
          IFLAG2=1;
        end
      end % end for K=1:NP
    if(IFLAG2 == 0)
      C(IR,IC) = C(IR,IC) + CE(J,L);
      end
    end  % end for L=1:3
  end   %end if(iflag1 == 0)
end  % end for J=1:3
end % end for I=1:NE
% ************************************************
%   THIRD STEP - SOLVE THE SYSTEM OF EQUATIONS
% ************************************************

V = inv(C)*B;
V=V';
% ************************************************
%   FOURTH STEP - OUTPUT THE RESULTS
% ************************************************
diary exam147.out
[ND, NE, NP]
[ [1:ND]' X' Y' V']
diary off
```

Figure 15.33 (Continued)

opment of the program basically involves four steps indicated in the program and explained as follows.

Step 1: This involves inputting the necessary data defining the problem. This is the only step that depends on the geometry of the problem at hand. Through a data file, we input the number of elements, the number of nodes, the number of fixed nodes, the prescribed values of the potentials at the free nodes, the x and y coordinates of all nodes, and a list identifying the nodes belonging to each element in the order of the local numbering 1-2-3. For the problem in Figure 15.32, the three sets of data for coordinates, element-node relationship, and prescribed potentials at fixed nodes are shown in Tables 15.3, 15.4, and 15.5, respectively.

TABLE 15.3 Nodal Coordinates of the Finite Element Mesh of Figure 15.32

Node	x	y	Node	x	y
1	0.0	0.0	12	0.0	0.4
2	0.2	0.0	13	0.2	0.4
3	0.4	0.0	14	0.4	0.4
4	0.6	0.0	15	0.6	0.4
5	0.8	0.0	16	0.0	0.6
6	1.0	0.0	17	0.2	0.6
7	0.0	0.2	18	0.4	0.6
8	0.2	0.2	19	0.0	0.8
9	0.4	0.2	20	0.2	0.8
10	0.6	0.2	21	0.0	1.0
11	0.8	0.2			

TABLE 15.4 Element-Node Identification

	Local Node No.				Local Node No.		
Element No.	1	2	3	Element No.	1	2	3
1	1	2	7	14	9	10	14
2	2	8	7	15	10	15	14
3	2	3	8	16	10	11	15
4	3	9	8	17	12	13	16
5	3	4	9	18	13	17	16
6	4	10	9	19	13	14	17
7	4	5	10	20	14	18	17
8	5	11	10	21	14	15	18
9	5	6	11	22	16	17	19
10	7	8	12	23	17	20	19
11	8	13	12	24	17	18	20
12	8	9	13	25	19	20	21
13	9	14	13				

TABLE 15.5 Prescribed Potentials at Fixed Nodes

Node No.	Prescribed Potential	Node No.	Prescribed Potential
1	0.0	18	100.0
2	0.0	20	100.0
3	0.0	21	50.0
4	0.0	19	0.0
5	0.0	16	0.0
6	50.0	12	0.0
11	100.0	7	0.0
15	100.0		

TABLE 15.6 Input Data for the Finite Element Program in Figure 15.33

```
NE = 25;
ND = 21;
NP = 15;
NL = [ 1 2 7
   2 8 7
   2 3 8
       3 9 8
       3 4 9
       4 10 9
       4 5 10
       5 11 10
       5 6 11
       7 8 12
      8 13 12
       8 9 13
       9 14 13
       9 10 14
       10 15 14
       10 11 15
       12 13 16
       13 17 16
       13 14 17
       14 18 17
       14 15 18
       16 17 19
       17 20 19
       17 18 20
       19 20 21];

   X = [0.0  0.2  0.4  0.6  0.8  1.0  0.0 ...
        0.2  0.4  0.6  0.8  0.0  0.2  0.4 ...
        0.6  0.0  0.2  0.4  0.0  0.2  0.0];

   Y = [0.0  0.0  0.0  0.0  0.0  0.0  0.2 ...
        0.2  0.2  0.2  0.2  0.4  0.4  0.4 ...
        0.4  0.6  0.6  0.6  0.8  0.8  1.0];

   NDP = [ 1 2 3 4 5 6 11 15 18 20 21 19 16 12 7];

   VAL = [ 0.0  0.0  0.0  0.0 0.0 ...
           50.0  100.0  100.0  100.0  100.0
           50.0  0.0   0.0   0.0   0.0];
```

Step 2: This step entails finding the element coefficient matrix $[C^{(e)}]$ for each element and the global coefficient matrix $[C]$. The procedure explained in the previous example is applied. Equation (15.6.4) can be written in general form as

$$\begin{bmatrix} 1 & 0 \\ 0 & C_{ff} \end{bmatrix} \begin{bmatrix} V_P \\ V_f \end{bmatrix} = \begin{bmatrix} 1 \\ -C_{fp} \end{bmatrix} [V_p]$$

or

$$[C]\,[V] = [B]$$

Both "global" matrix $[C]$ and matrix $[B]$ are calculated at this stage.

Step 3: The global matrix obtained in the previous step is inverted. The values of the potentials at all nodes are obtained by matrix multiplication as in eq. (15.76b). Instead of inverting the global matrix, it is also possible to solve for the potentials at the nodes using Gaussian elimination technique.

Step 4: This involves outputting the result of the computation.

The input and output data are presented in Tables 15.6 and 15.7, respectively.

TABLE 15.7 Output Data of the Program in Figure 15.33

Node	X	Y	Potential
1	0.00	0.00	0.000
2	0.20	0.00	0.000
3	0.40	0.00	0.000
4	0.60	0.00	0.000
5	0.80	0.00	0.000
6	1.00	0.00	50.000
7	0.00	0.20	0.000
8	0.20	0.20	18.182
9	0.40	0.20	36.364
10	0.60	0.20	59.091
11	0.80	0.20	100.000
12	0.00	0.40	0.000
13	0.20	0.40	36.364
14	0.40	0.40	68.182
15	0.60	0.40	100.000
16	0.00	0.60	0.000
17	0.20	0.60	59.091
18	0.40	0.60	100.000
19	0.00	0.80	0.000
20	0.20	0.80	100.000
21	0.00	1.00	50.000

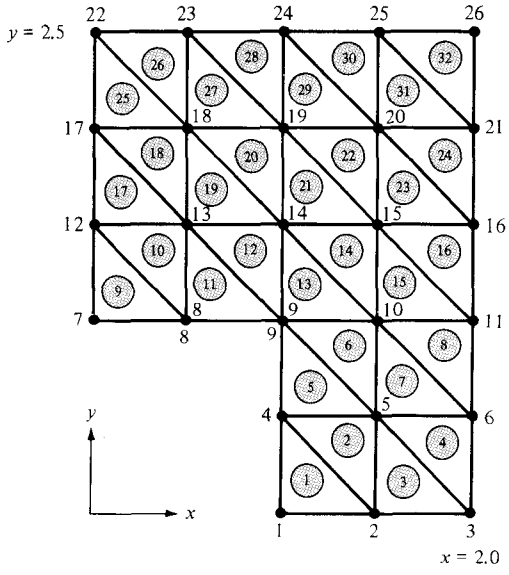


Figure 15.34 For Practice Exercise 15.7.

PRACTICE EXERCISE 15.7

Rework Example 15.3 using the finite element method. Divide the solution region into triangular elements as shown in Figure 15.34. Compare the solution with that obtained in Example 15.3 using the finite difference method.

Answer: See Example 15.3.

SUMMARY

1. Electric field lines and equipotential lines due to coplanar point sources can be plotted using the numerical technique presented in this chapter. The basic concept can be extended to plotting magnetic field lines.
2. An EM problem in the form of a partial differential equation can be solved using the finite difference method. The finite difference equation that approximates the differential equation is applied at grid points spaced in an ordered manner over the whole solution region. The field quantity at the free points is determined using a suitable method.
3. An EM problem in the form of an integral equation is conveniently solved using the moment method. The unknown quantity under the integral sign is determined by matching both sides of the integral equation at a finite number of points in the domain of the quantity.
4. While the finite difference method is restricted to problems with regularly shaped solution regions, the finite element method can handle problems with complex geometries. This method involves dividing the solution region into finite elements, deriving equations for a typical element, assembling all elements in the region, and solving the resulting system of equations.

Typical examples on how to apply each method to some practical problems have been shown. Computer programs for solving the problems are provided wherever needed.

REVIEW QUESTIONS

15.1 At the point (1, 2, 0) in an electric field due to coplanar point charges, $\mathbf{E} = 0.3\,\mathbf{a}_x - 0.4\,\mathbf{a}_y$ V/m. A differential displacement of 0.05 m on an equipotential line at that point will lead to point

(a) (1.04, 2.03, 0)
(b) (0.96, 1.97, 0)
(c) (1.04, 1.97, 0)
(d) (0.96, 2.03, 0)

15.2 Which of the following is *not* a correct finite difference approximation to dV/dx at x_o if $h = \Delta x$?

(a) $\dfrac{V(x_o + h) - V(x_o)}{h}$

(b) $\dfrac{V(x_o) - V(x_o - h)}{h}$

(c) $\dfrac{V(x_o + h) - V(x_o - h)}{h}$

(d) $\dfrac{V(x_o + h) - V(x_o - h)}{2h}$

(e) $\dfrac{V(x_o + h/2) - V(x_o - h/2)}{h}$

15.3 The triangular element of Figure 15.35 is in free space. The approximate value of the potential at the center of the triangle is

(a) 10 V
(b) 7.5 V
(c) 5 V
(d) 0 V

15.4 For finite difference analysis, a rectangular plate measuring 10 by 20 cm is divided into eight subregions by lines 5 cm apart parallel to the edges of the plates. How many free nodes are there if the edges are connected to some source?

(a) 15
(b) 12
(c) 9

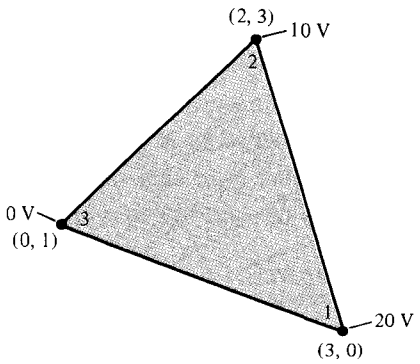


Figure 15.35 For Review Questions 15.3 and 15.10.

(d) 6

(e) 3

15.5 Using the difference equation $V_n = V_{n-1} + V_{n+1}$ with $V_0 = V_5 = 1$ and starting with initial values $V_n = 0$ for $1 \leq n \leq 4$, the value of V_2 after the third iteration is

(a) 1

(b) 3

(c) 9

(d) 15

(e) 25

15.6 The coefficient matrix $[A]$ obtained in the moment method does *not* have one of these properties:

(a) It is dense (i.e., has many nonzero terms).

(b) It is banded.

(c) It is square and symmetric.

(d) It depends on the geometry of the given problem.

15.7 A major difference between the finite difference and the finite element methods is that

(a) Using one, a sparse matrix results in the solution.

(b) In one, the solution is known at all points in the domain.

(c) One applies to solving partial differential equation.

(d) One is limited to time-invariant problems.

15.8 If the plate of Review Question 14.4 is to be discretized for finite element analysis such that we have the same number of grid points, how many triangular elements are there?

(a) 32

(b) 16

(c) 12

(d) 9

15.9 Which of these statements is *not* true about shape functions?

(a) They are interpolatory in nature.
(b) They must be continuous across the elements.
(c) Their sum is identically equal to unity at every point within the element.
(d) The shape function associated with a given node vanishes at any other node.
(e) The shape function associated with a node is zero at that node.

15.10 The area of the element in Figure 15.35 is

(a) 14
(b) 8
(c) 7
(d) 4

Answers: 15.1a, 15.2c,[4] 15.3a, 15.4e, 15.5c, 15.6b, 15.7a, 15.8b, 15.9e, 15.10d.

PROBLEMS

15.1 Using the program developed in Example 15.1 or your own equivalent code, plot the electric field lines and equipotential lines for the following cases:

(a) Three point charges -1, 2, and 1 C placed at $(-1, 0)$, (0,2), and (1,0), respectively.
(b) Five identical point charges 1 C located at $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$, and $(0, 0)$, respectively.

15.2 Given the one-dimensional differential equation

$$\frac{d^2y}{dx^2} = 0, \qquad 0 \leq x \leq 1$$

subject to $y(0) = 0$, $y(1) = 10$, use the finite difference (iterative) method to find $y(0.25)$. You may take $\Delta = 0.25$ and perform 5 iterations.

15.3 (a) From the table below, obtain $\frac{dV}{dx}$ and $\frac{d^2V}{dx^2}$ at $x = 0.15$.

x	0.1	0.15	0.2	0.25	0.3
V	1.0017	1.5056	2.0134	2.5261	3.0452

(b) The data in the table above are obtained from $V = 10 \sinh x$. Compare your result in part (a) with the exact values.

[4]The formula in (a) is known as a forward-difference formula, that in (b) as a backward-difference formula, and that in (d) or (e) as a central-difference formula.

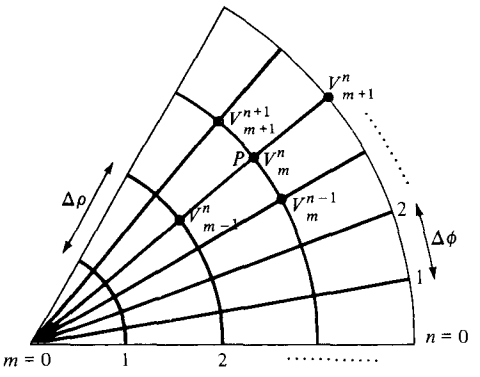


Figure 15.36 Finite difference grid in cylindrical coordinates; for Problem 15.5.

15.4 Show that the finite difference equation for Laplace's equation in cylindrical coordinates, $V = V(\rho, z)$, is

$$V(\rho_o, z_o) = \frac{1}{4}\left[V(\rho_o, z_o + h) + V(\rho_o, z_o - h) + \left(1 + \frac{h}{2\rho_o}\right) V(\rho_o + h, z_o) + \left(1 - \frac{h}{2\rho_o}\right) V(\rho_o - h, z_o)\right]$$

where $h = \Delta z = \Delta\rho$.

15.5 Using the finite difference representation in cylindrical coordinates (ρ, ϕ) at a grid point P shown in Figure 15.36, let $\rho = m\,\Delta\rho$ and $\phi = n\,\Delta\phi$ so that $V(\rho, \phi)|_P = V(m\Delta\rho, n\Delta\phi) = V^n_m$. Show that

$$\nabla^2 V|_{m,n} = \frac{1}{\Delta\rho^2}\left[\left(1 - \frac{1}{2m}\right) V^n_{m-1} - 2V^n_m + \left(1 + \frac{1}{2m}\right) V^n_{m+1} + \frac{1}{(m\,\Delta\phi)^2}(V^{n-1}_m - 2\,V^n_m + V^{n+1}_m)\right]$$

15.6 A square conducting trough has its four sides held at potentials -10, 0, 30, and 60 V. Determine the potential at the center of the trough.

15.7 Use FDM to calculate the potentials at nodes 1 and 2 in the potential system shown in Figure 15.37.

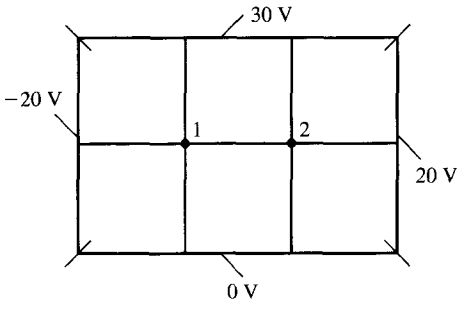


Figure 15.37 For Problem 15.7.

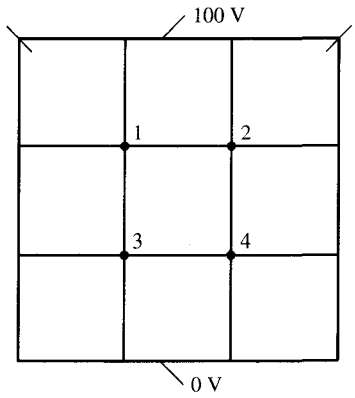


Figure 15.38 For Problem 15.9.

15.8 Rework Problem 15.7 if $\rho_S = \dfrac{100}{\pi}$ nC/m^2, $h = 0.1$ m, and $\varepsilon = \varepsilon_o$, where h is the mesh size.

15.9 Consider the potential system shown in Figure 15.38. (a) Set the initial values at the free nodes equal to zero and calculate the potential at the free nodes for five iterations. (b) Solve the problem by the band matrix method and compare result with part (a).

15.10 Apply the band matrix technique to set up a system of simultaneous difference equations for each of the problems in Figure 15.39. Obtain matrices $[A]$ and $[B]$.

15.11 (a) How would you modify matrices $[A]$ and $[B]$ of Example 15.3 if the solution region had charge density ρ_S?

(b) Write a program to solve for the potentials at the grid points shown in Figure 15.40 assuming a charge density $\rho_S = x(y - 1)$ nC/m^2. Use the iterative finite difference method and take $\varepsilon_r = 1.0$.

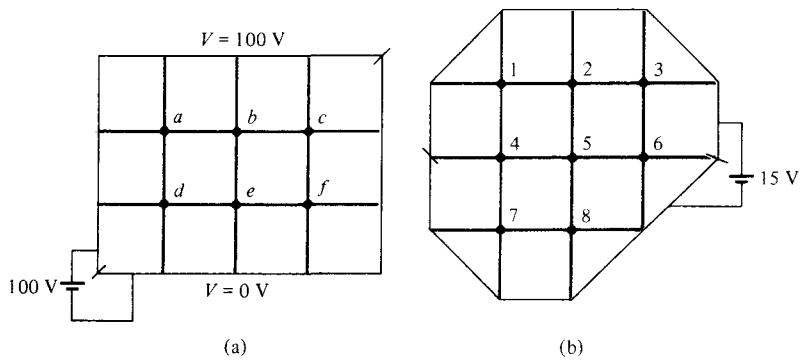


Figure 15.39 For Problem 15.10.

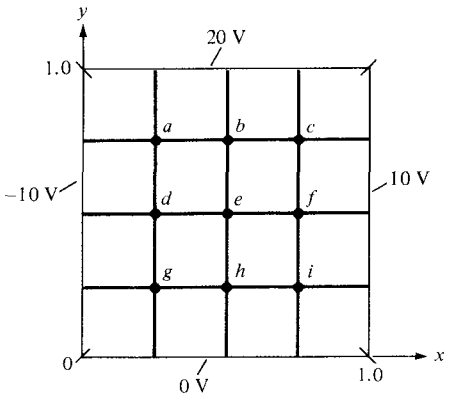


Figure 15.40 For Problem 15.11.

15.12 The two-dimensional wave equation is given by

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

By letting $\Phi^j_{m,n}$ denote the finite difference approximation of $\Phi(x_m, z_n, t_j)$, show that the finite difference scheme for the wave equation is

$$\Phi^{j+1}_{m,n} = 2\,\Phi^j_{m,n} - \Phi^{j-1}_{m,n} + \alpha\,(\Phi^j_{m+1,n} + \Phi^j_{m-1,n} - 2\,\Phi^j_{m,n}) + \alpha\,(\Phi^j_{m,n+1} + \Phi^j_{m,n-1} - 2\,\Phi^j_{m,n})$$

where $h = \Delta x = \Delta z$ and $\alpha = (c\Delta t/h)^2$.

15.13 Write a program that uses the finite difference scheme to solve the one-dimensional wave equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2}, \qquad 0 \le x \le 1, \qquad t > 0$$

given boundary conditions $V(0, t) = 0$, $V(1, t) = 0$, $t > 0$ and the initial condition $\partial V/\partial t\,(x, 0) = 0$, $V(x, 0) = \sin \pi x$, $0 < x < 1$. Take $\Delta x = \Delta t = 0.1$. Compare your solution with the exact solution $V(x, t) = \sin \pi x \cos \pi t$ for $0 < t < 4$.

15.14 (a) Show that the finite difference representation of Laplace's equation using the nine-node molecule of Figure 15.41 is

$$V_o = 1/8\,(V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8)$$

(b) Using this scheme, rework Example 15.4.

15.15 A transmission line consists of two identical wires of radius a, separated by distance d as shown in Figure 15.42. Maintain one wire at 1 V and the other at -1 V and use MOM to find the capacitance per unit length. Compare your result with exact formula for C in Table 11.1. Take $a = 5$ mm, $d =$ cm, $\ell = 5$ m, and $\epsilon = \epsilon_o$.

15.16 Determine the potential and electric field at point $(-1, 4, 5)$ due to the thin conducting wire of Figure 15.19. Take $V_o = 1$ V, $L = 1$ m, $a = 1$ mm.

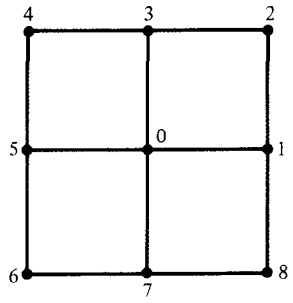


Figure 15.41 Nine-node molecule of Problem 15.14.

15.17 Two conducting wires of equal length L and radius a are separated by a small gap and inclined at an angle θ as shown in Figure 15.43. Find the capacitance between the wires using the method of moments for cases $\theta = 10°, 20°, \ldots, 180°$. Take the gap as 2 mm, $a = 1$ mm, $L = 2$ m, $\varepsilon_r = 1$.

15.18 Given an infinitely long thin strip transmission line shown in Figure 15.44(a), we want to determine the characteristic impedance of the line using the moment method. We divide each strip into N subareas as in Figure 15.44(b) so that on subarea i,

$$V_i = \sum_{j=1}^{2N} A_{ij}\,\rho_j$$

where

$$A_{ij} = \begin{cases} \dfrac{-\Delta\ell}{2\pi\varepsilon_o} \ln R_{ij}, & i \neq j \\ \dfrac{-\Delta\ell}{2\pi\varepsilon_o} [\ln \Delta\ell - 1.5], & i = j \end{cases}$$

R_{ij} is the distance between the ith and jth subareas, and $V_i = 1$ or -1 depending on whether the ith subarea is on strip 1 or 2, respectively. Write a program to find the characteristic impedance of the line using the fact that

$$Z_o = \frac{\sqrt{\mu_o \varepsilon_o}}{C}$$

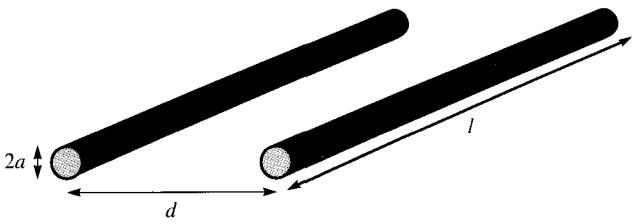
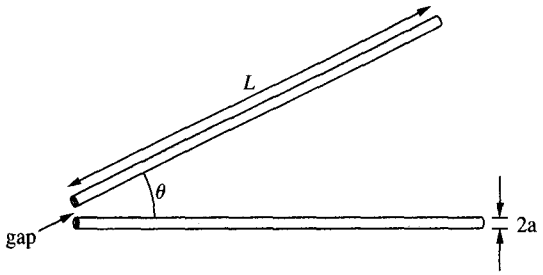


Figure 15.42 For Problem 15.15.

Figure 15.43 For Problem 15.17.



where C is the capacitance per unit length and

$$C = \frac{Q}{V_d} = \frac{\sum_{i=1}^{N} \rho_i \Delta\ell}{V_d}$$

and $V_d = 2$ V is the potential difference between strips. Take $H = 2$ m, $W = 5$ m, and $N = 20$.

15.19 Consider the coaxial line of arbitrary cross section shown in Figure 15.45(a). Using the moment method to find the capacitance C per length involves dividing each conductor into N strips so that the potential on the jth strip is given by

$$V_j = \sum_{i=1}^{2N} \rho_i A_{ij}$$

where

$$A_{ij} = \begin{cases} \dfrac{-\Delta\ell}{2\pi\varepsilon} \ln \dfrac{R_{ij}}{r_o}, & i \neq j \\ \dfrac{-\Delta\ell}{2\pi\varepsilon} \left[\ln \dfrac{\Delta\ell_i}{r_o} - 1.5 \right], & i = j \end{cases}$$

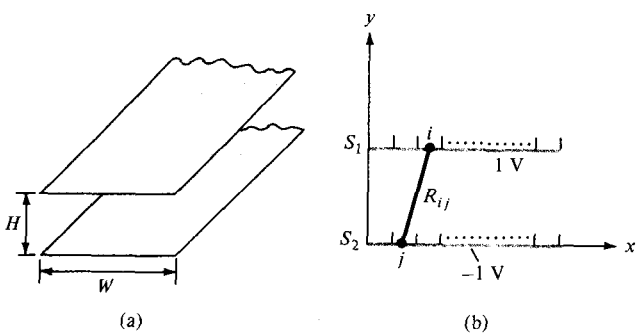


Figure 15.44 Analysis of strip transmission line using moment method; for Problem 15.18.

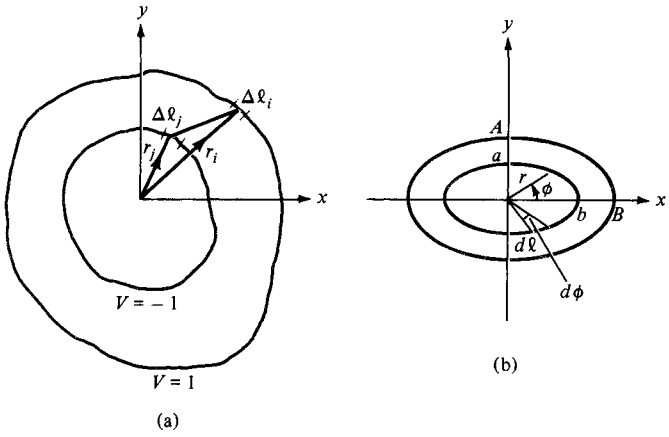


Figure 15.45 For Problem 15.19; coaxial line of **(a)** arbitrary cross section, and **(b)** elliptical cylindrical cross section.

and $V_j = -1$ or 1 depending on whether $\Delta\ell_i$ lies on the inner or outer conductor respectively. Write a Matlab program to determine the total charge per length on a coaxial cable of elliptical cylindrical cross section shown in Figure 15.45(b) using

$$Q = \sum_{i=1}^{N} \rho_i$$

and the capacitance per unit length using $C = Q/2$.

(a) As a way of checking your program, take $A = B = 2$ cm and $a = b = 1$ cm (coaxial line with circular cross section), and compare your result with the exact value of $C = 2\pi\varepsilon/\ln(A/a)$.

(b) Take $A = 2$ cm, $B = 4$ cm, $a = 1$ cm, and $b = 2$ cm.
(*Hint:* For the inner ellipse of Figure 15.45(b), for example,

$$r = \frac{a}{\sqrt{\sin^2\phi + v^2\cos^2\phi}}$$

where $v = a/b$, $d\ell = r\,d\phi$. Take $r_o = 1$ cm.)

15.20 The conducting bar of rectangular cross section is shown in Figure 15.46. By dividing the bar into N equal segments, we obtain the potential at the jth segment as

$$V_j = \sum_{i=1}^{N} q_i A_{ij}$$

where

$$A_{ij} = \begin{cases} \dfrac{1}{4\pi\varepsilon_o R_{ij}}, & i \neq j \\ \dfrac{1}{2\varepsilon_o\sqrt{\pi h\Delta}}, & i = j \end{cases}$$

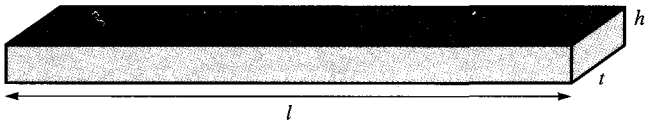


Figure 15.46 For Problem 15.20.

and Δ is the length of the segment. If we maintain the bar at 10 V, we obtain

$$[A][q] = 10[I]$$

where $[I] = [1\ 1\ 1.\ .\ .1]^T$ and $q_i = \rho_v th\Delta$.

(a) Write a program to find the charge distribution ρ_v on the bar and take $\ell = 2$ m, $h = 2$ cm, $t = 1$ cm, and $N = 20$.

(b) Compute the capacitance of the isolated conductor using

$$C = Q/V = (q_1 + q_2 + .\ .\ . + q_N)/10$$

15.21 Another way of defining the shape functions at an arbitrary point (x, y) in a finite element is using the areas A_1, A_2, and A_3 shown in Figure 15.47. Show that

$$\alpha_k = \frac{A_k}{A}, \qquad k = 1, 2, 3$$

where $A = A_1 + A_2 + A_3$ is the total area of the triangular element.

15.22 For each of the triangular elements of Figure 15.48,

(a) Calculate the shape functions.

(b) Determine the coefficient matrix.

15.23 The nodal potential values for the triangular element of Figure 15.49 are $V_1 = 100$ V, $V_2 = 50$ V, and $V_3 = 30$ V. (a) Determine where the 80 V equipotential line intersects the boundaries of the element. (b) Calculate the potential of (2, 1).

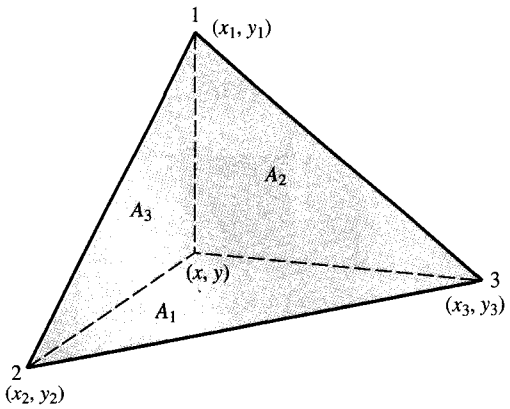


Figure 15.47 For Problem 15.21.

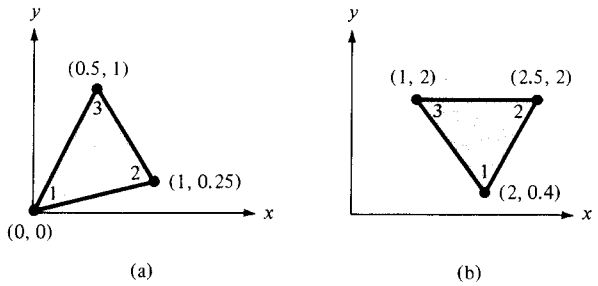


Figure 15.48 Triangular elements of Problem 15.22.

15.24 The triangular element shown in Figure 15.50 is part of a finite element mesh. If $V_1 = 8$ V, $V_2 = 12$ V, and $V_3 = 10$ V, find the potential at (a) (1,2) and (b) the center of the element.

15.25 Determine the global coefficient matrix for the two-element region shown in Figure 15.51.

15.26 Find the global coefficient matrix of the two-element mesh of Figure 15.52.

15.27 For the two-element mesh of Figure 15.52, let $V_1 = 10$ V and $V_3 = 30$ V. Find V_2 and V_4.

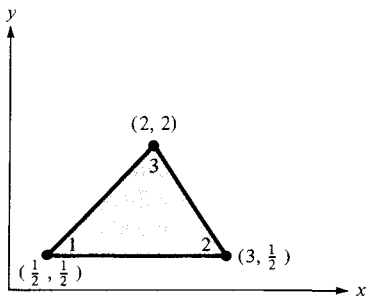


Figure 15.49 For Problem 15.23.

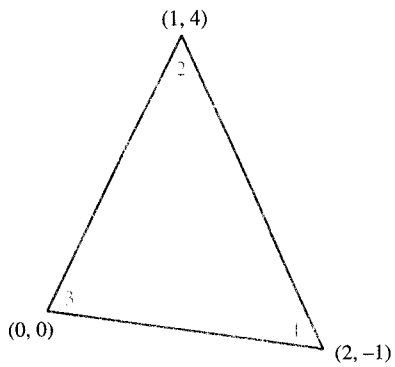


Figure 15.50 For Problem 15.24.

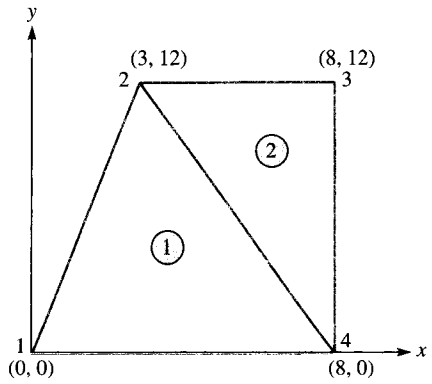


Figure 15.51 For Problem 15.25.

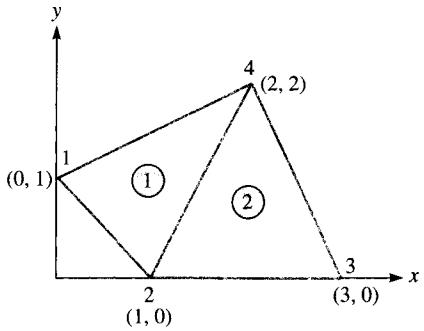


Figure 15.52 For Problem 15.26 and 15.27.

15.28 The mesh in Figure 15.53 is part of a large mesh. The shading region is conducting and has no elements. Find $C_{5,5}$ and $C_{5,1}$.

15.29 Use the program in Figure 15.33 to solve Laplace's equation in the problem shown in Figure 15.54 where $V_o = 100$ V. Compare the finite element solution to the exact solution in Example 6.5; that is,

$$V(x, y) = \frac{4V_o}{\pi} \sum_{k=0}^{\infty} \frac{\sin n\pi x \sinh n\pi y}{n \sinh n\pi}, \qquad n = 2k + 1$$

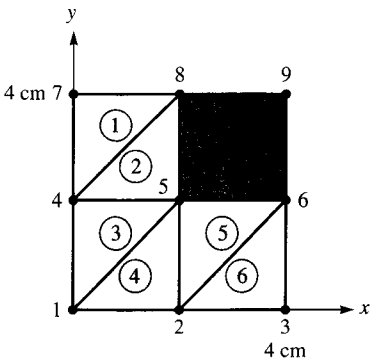


Figure 15.53 For Problem 15.28.

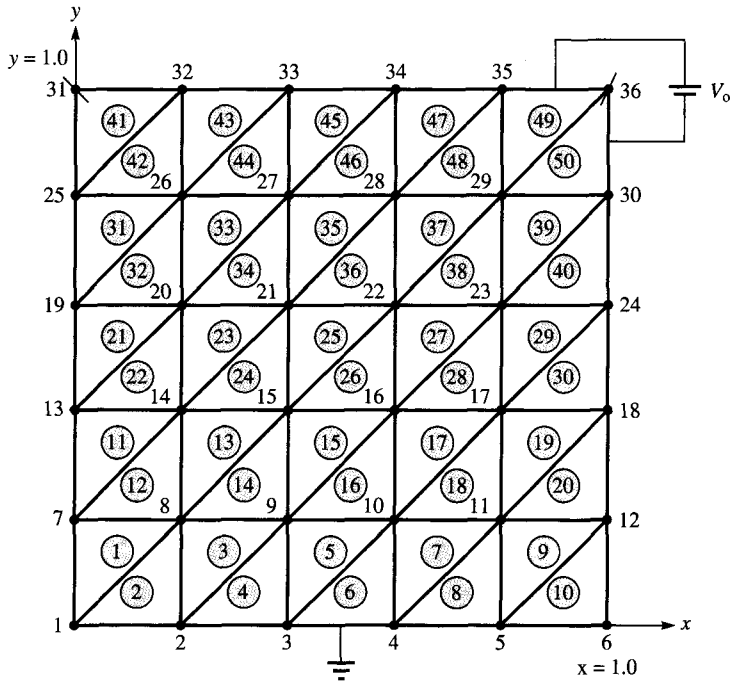


Figure 15.54 For Problem 15.29.

15.30 Repeat the preceding problem for $V_o = 100 \sin \pi x$. Compare the finite element solution with the theoretical solution [similar to Example 6.6(a)]; that is,

$$V(x, y) = \frac{100 \sin \pi x \sinh \pi y}{\sinh \pi}$$

15.31 Show that when a square mesh is used in FDM, we obtain the same result in FEM when the squares are cut into triangles.